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Intersection Properties of Balls in Spaces of Vector-valued Function

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1. Introduction

A closed subspace M of a Banach space E is said to have the n-ball (strong n-ball) property, whenever $B_1, \cdots, B_n$ are closed balls in E with $M \cap B_i \neq \emptyset$ for each i, and $int \bigcap_{i=1}^{n} B_i \neq \emptyset$ ($\bigcap_{i=1}^{n} B_i \neq \emptyset$), then we have $M \cap \bigcap_{i=1}^{n} B_i \neq \emptyset$.

Alfsen and Effros [1] showed that a subspace with 3-ball property actually has the n-ball roperty for all n, and they called such subspaces M-ideals.

Why should this definition be less natural that the previous definition? The main reason is ne following useful characterization of M-ideals: M is an M-ideal in E iff its annihilator, M^0, s an L-summand in E^*. This definition does not characterize subspaces with the strong n-ball roperty. It is not even whether the strong 3-ball property implies the strong n-ball property or higher values of n.

To obtain the duality theorem, it is necessary to assume interior intersection. But why hould we assume $\bigcap_{i=1}^{n} int\, B_i \neq \emptyset$? Why not assume that $M \cap int\, B_i \neq \emptyset$? This leads us to the ossibility of defining the n-ball property in several, not necessarily epuivalent, ways.

In this note we examines the relationship between these definitions. We give positive results wherever possible, and counterexamples otherwise.

In section 2, we introduce some preliminaries which are necessary to explain the theorems.

In section 3, we will give the main theorems. In the following, Banach space is denoted E, closed balls in E are denoted $B(x_1, r_1) = B_1, \cdots, B(x_n, r_n) = B_n$. The dual space of E is written E^*. If X is a compact, Hausdorff space, let $C(X, E)$ denote the supnormed Banach space of continuous functions from X to E.

The terminology and notation used in this note will be adopted [3], [8], and [14].

2. Preliminaries

Definition 2.1. Let M be a subspace of the Banach space E and $n \in N$.

(1) We say that E has the *n-ball property for open balls* if for every family $V(x_1, r_1), \cdots, V(x_n, r_n)$ of open balls such that $\bigcap_{i=1}^{n} V(x_i, r_i) \neq \emptyset$ and $M \cap V(x_i, r_i) \neq \emptyset$ (all $i \in \{1, \cdots, n\}$) the intersetion $M \cap \bigcap_{i=1}^{n} V(x_i, r_i)$ is non-empty.

(2) M is said to have the *n-ball property for closed balls* if $M \cap \bigcap_{i=1}^{n} B(x_i, r_i) \neq \emptyset$ for every family $B(x_1, r_1), \cdots, B(x_n, r_n)$ of closed balls ($B(x, r) = \{y \mid y \in E, \|x-y\| \leqq r$ for $x \in E$ and $r \geqq 0$)

such that $int \bigcap_{i=1}^{n} B(x_i, r_i) \neq \emptyset$ and $M \cap B(x_i, r_i) \neq \emptyset$ (all $i \in \{1, \cdots, n\}$).

Definition 2.2. Let E de a Banach space and a closed subspace M of E has *strong n-ball property* if, for all closed balls $B_1, \cdots, B_n$, the assumptions $M \cap B_i \neq \emptyset$ for each i, and $\bigcap_{i=1}^{n} B_i \neq \emptyset$ imply that $M \cap \bigcap_{i=1}^{n} B_i \neq \emptyset$.

Definition 2.3. Let E be a Banach space and M a closed subspace of E. M is called an *M-ideal* if M^0, the annihilator of M in E^*, is an L-summand of E^*. This means that there is a projection $P: E^* \to M^0$ satisfying the identity $\|f\| = \|pf\| + \|f - pf\|$.

Definition 2.4. Let E be a Banach space and a closed subspace M of E has the *medium n-ball property* in E if, for all closed balls $B_1, \cdots, B_n$, the assumptions $\bigcap_{i=1}^{n} B_i \neq \emptyset$ and $M \cap int B_i \neq \emptyset$ for each i, imply that $M \cap \bigcap_{i=1}^{n} B_i \neq \emptyset$.

Theorem 2.5. *Let M be a closed subspace of the Banach space E. Then the following are equivalent:*

(1) *M is an M-ideal.*

(2) *M has the 3-ball property for open balls.*

(3) *M has the n-ball property for open balls for every $n \in N$.*

Proof. The proof may be found in [1].

Proposition 2.6. *Let $V(x_i, r_i)$, $i = 1, \cdots, n$ be a family of n open balls in the Banach space E such that there is an X_0 in $\bigcap_{i=1}^{n} V(x_i, r_i)$. Then there exists a δ in $(0, 1)$ such that, for ε in $(0, 1]$ and x in $\bigcap_{i=1}^{n} V(x_i, r_i + \varepsilon)$, the intersection $V(x, \delta\varepsilon) \cap \bigcap_{i=1}^{n} V(x_i, r_i + \delta\varepsilon)$ is non-empty (we may take $\delta = (1 + \frac{m}{2M+1})^{-1}$, where $M = max\{r_i \mid i = 1, \cdots, n\}$, $m = min\{r - \|x_i - x_0\| \mid i = 1, \cdots, n\}$).*

Proof. The proof may be found in [3].

Proposition 2.7. *Let M be a closed subspace of the Banach space E and $n \in N$.*

(1) *If M satisfies the $(n+1)$-ball property for open balls, then M satisfies the n-ball property for closed balls.*

(2) *The n-ball property for closed balls implies the n-ball property open balls.*

Proof. The proof may be found in [3].

Theorem 2.8. *Let M be a closed subspace of the Banach space E and $\nu: E \to E/M$ the canonical mapping onto the quotient. The following are equivalent:*

(1) *M is an M-ideal.*

(2) *M satisfies the 3-ball property for open balls.*

(3) *M satisfies the n-ball property for open balls (all $n \in N$).*

(4) *M satisfies the 3-ball property for closed balls.*

(5) *M satisfies the n-ball property for closed balls (all $n \in N$).*

(6) *If $V_1, \cdots, V_n$ are open balls such that $\bigcap_{i=1}^{n} V_i \neq \emptyset$, then $\nu(\bigcap_i B_i) = \bigcap_i \nu(B_i)$.*

(7) *If $B_1, \cdots, B_n$ are closed balls such that int* $\bigcap_{i=1}^{n} B_i \neq \emptyset$, *then* $\nu(\bigcap_i B_i) = \bigcap_i \nu(B_i)$

Proof. (1)～(5) are equivalent by th. 2. 5. and prop. 2. 7. The equivalences (3)⇔(6) and)⇔(7) are easily verified.

The following is the natural definition to work with in order to prove the duality theorem.

finition 2. 9. Let E be a Banach space and a closed subspace M of E has the *weak n-ball operty* in E if the conditions *int* $\bigcap_{i=1}^{n} B_i \neq \emptyset$ *and* $M \cap$ *int* $B_i \neq \emptyset$ for each i, imply that $\cap \bigcap_{i=1}^{n} B_i \neq \emptyset$.

efinition 2. 10. Let $\{x_1, x_2, \cdots\} \subset E$ and $\{f_1, f_2, \cdots\} \subset E^*$. We say that (x_n, f_n) is a *Markusěvic sis* for E if $f_m(x_n) = \delta_{mn}$ the linear span of $\{x_n\}$ is dence in E, and $\{f_n\}$ separates points of E.

efinition 2. 11. A Banach space E is *strictly convex* if every norm one vector is an extreme oint of the unit ball.

efinition 2. 12. A Banach space E is *locally uniformly convex* if the conditions $\|x_n\| \to \|x\| = 1$ nd $\|x_n + x\| \to 2$ imply that $x_n \to x$.

efinition 2. 13. A Banach space E is *uniformly convex* if the conditions $\|x_n\| = \|y_n\| = 1$ and $x_n + y_n\| \to 2$ imply that $x_n - y_n \to 0$.

3. Main Theorems

It will evident that the following fails if we do not assume interior intersection.

roposition 3. 1. *Suppose that M has the weak $(n+1)$-ball property in E. Then M has the -ball property in E.*

Proof. The proof may be found in [10].

orollary 3. 2. *The weak n-ball property is equivalent to the n-ball property.*

Proof. The proof may be found in [14], [15].

We will show that the n-ball property does not imply the medium n-ball property, and that he medium n-ball property does not imply the strong n-ball property.

xample 3. 3. *The n-ball property does not imply the medium n-ball property.*

Proof. The proof may be found in [3], [7], [10] and [13].

roposition 3. 4. *For each n, Y^0 has the medium n-ball property in $C(X, E)$.*

Proof. Suppose that $f \in \bigcap_{i=1}^{n} B(f_i, r_i)$ and that $Y^0 \cap$ *int* $B(f_i, r_i) \neq \emptyset$ for each i. Then $\|f_i(y)\| < r_i$, for each $y \in Y$, $i \leqq n$. Let $Z = \{x \in X \mid \|f_i(x)\| \geqq r_i$, for at least one $i\}$. Then Z is closed, and disjoint from Y.

We may suppose that $Z \neq \emptyset$, since otherwise $0 \in Y^0 \cap \bigcap_{i=1}^{n} B(f_i, r_i)$. Then there is a continuous function $h: X \to [0, 1]$ with $h(Y) = \{0\}$ and $h(Z) = \{1\}$.

Define $g \in Y^0$ by $g(x) = h(x) f(x)$, for $x \in X$.

We claim that $g\in\bigcap_{i=1}^{n} B(f_i, r_i)$. First note that $f(x)\in\bigcap_{i=1}^{n} B(f_i(x), r_i)$ for each $x\in X$. If $x\in Z$, then $g(x)=f(x)\in\bigcap_{i=1}^{n} B(f_i(x), r_i)$, as required. If $x\in Z$, then by definition $0\in\bigcap_{i=1}^{n} B(f_i(x), r_i)$, and so again we have $g(x)=h(x)f(x)+(1-h(x))$, $0\in\bigcap_{i=1}^{n} B(f_i(x), r_i)$.

Theorem 3.5. *Let X be a space in which sequences suffice (e.g. a metric space). Suppose that the set of extreme points of the unit ball of E is not closed. Then the following are equivalent:*

(1) *Y is clopen*

(2) *Y^0 has the strong n-ball property, for all n.*

(3) *Y^0 has the strong 2-ball property in $C(X, E)$.*

Proof. (1)$\Longrightarrow$(2). Let $Z=X/Y$. If Z is closed then $C(X, E)$ is the direct sum of Y and Z, and $\|f+g\|=\max\{\|f\|, \|g\|\}$, for $f\in Y^0$, $g\in Z^0$. Easy calculation establish the strong n-ball property.

(2)$\Longrightarrow$(3). Trivial.

(3)$\Longrightarrow$(1). By hypothesis, there exist extreme points a of the unit ball of E, and $a, b\in E$ with $a_n\to a$, $\|a\pm b\|\leqq 1$, but $b\neq 0$. Suppose that Y is not clopen; we will show that Y^0 fails the strong 2-ball property.

Now we can find $x_n\notin Y$ with $x_n\to y\in Y$. Define a continuous map $\phi: X\to[0,1]$ by setting $\phi(Y)=\{0|\phi(x_n)=1/n\}$; and then extending by Tietze's theorem.

Define h_+, h_-, $C([0,1], E)$ by $h_\pm(1/n)=b\pm a$, $h_\pm(0)=b\pm a$, and by linear interpolation elsewhere. Let $f=h_+\circ\phi, g=h_-\circ\phi$, and consider the balls $B(f,1)$ and $B(g,1)$. It is routine to verify that $\frac{1}{2}(f+g)\in B(f,1)\cap B(g,1)$, that $f-b-a\in Y^0\cap B(f,1)$ and that $g-b+a\in Y^0\cap B(g,1)$. But if $h\in B(f,1)\cap B(g,1)$ then

$$\begin{aligned} h(x_n) &\in B(f(x_n),1)\cap B(g(x_n),1)\\ &=B(b+a_n,1)\cap B(b-a_n,1)\\ &=\{b\},\end{aligned}$$

whence $h(Y)=\lim_{n\to\infty} h(x_n)=b=0$. Thus $B(f,1)\cap B(g,1)$ does not meet Y.

Corollary 3.6. *The medium n-ball property does not imply the strong n-ball property.*

Proof. The proof may be found in [14].

Theorem 3.7. *Suppose that E is strictly convex (i.e. every norm one vector is an extreme point of the unit ball). Then Y^0 has the strong n-ball property in $C(X, E)$, for every n.*

Proof. Suppose that $f\in\bigcap_{i=1}^{n} B(f_i, r_i)$ and that $Y^0\cap B(f_i, r_i)\neq\emptyset$, for $i\leqq n$. Define a set valued map $\phi: X/2^E\to\phi$ by $\phi(x)=\bigcap_{i=1}^{n} B(f_i(x), r_i)$. Clearly each $\phi(x)$ is closed and convex; we claim that ϕ is lower semi-continuous. If K is a closed subset of E, we must show that $\{x\in X|\phi(x)\subseteq K\}$ is closed. So let $x_a\to x$ in X, and suppose $\phi(x_a)\subseteq K$. We consider two cases, if $\phi(x)=\{f(x)\}$ is a singleton, then

$$f(x)\leftarrow f(x_a)\in\phi(x_a)\subseteq K, \text{ and so } \phi(x)\subseteq K.$$

Otherwise, suppose $\phi(x)$ contains two distinct points, a and b. For the time being, fix $\lambda\in(0,1)$. Now

$$\|a-f_i(x)\| \leqq r_i \text{ and } \|b-f_i(x)\| \leqq r_i, \text{ for } i=1,2,\cdots,n.$$

strict convexity, $\|\lambda a+(1-\lambda)b-f_i(x)\|<r_i$, for each i. Hence $\|\lambda a+(1-\lambda)b-f_i(x_\alpha)\|<r_i$, r all sufficiently large α. But then $\lambda a+(1-\lambda)b \in \phi(x_\alpha) \subseteq K$. Since K is closed, it follows that $b \in K$. This proves that $\phi(x) \subseteq K$, as required. Clearly $0 \in \phi(y)$ whenever $y \in Y$. Define $: X \to 2^E$ by $\phi_0(x)=\phi(x)$, for $x \notin Y$, and by $\phi_0(x)=\{0\}$, for $x \in Y$. It is routine to verify at is lower semicontinuous. By Michael's selection theorem [12] there is a continuous map $: X \to E$ with $g(x) \in \phi_0(x)$ for all $x \in X$. Clearly $g \in Y^0 \cap \bigcap_{i=1}^{n} B(f_i, r_i)$. We remark that if E is rictly convex, then no nontrivial subspace of E has ever the 2-ball property. To see this, ppose $\{0\} \subsetneqq M \subsetneqq E$, and choose $x \in E$ with $d(x, M)=1$.

By adding an element of M, we may suppose $|x|>1$. By [15, theorem 3], we may suppose at $M \cap B(x,1) \neq \emptyset$. By strict convexity, $M \cap B(x,1)$ contains only one point. Let us write $M \cap B(x,1)=\{y\}$. Again by strict convexity, $\|x\|<\|y\|+\|x-y\|=\|y\|+1$. Choose r so that $\|x\|-1<r<\|y\|$.

Then $B(0,r) \cap B(x,1)$ has non-empty interior, $M \cap B(0,r)$ is obviously non-empty, but $I \cap B(x,1)$ is non-empty and disjoint from $B(0,r)$. Thus theorem 3.7 say that if E is strictly onvex, then every M-ideal in $C(x,E)$ has the strong n-ball property for all n. Even for the pecial case $E=C$, this seems to be new.

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