On the Mystery of Geometrical Figures

Yoshimasa MICHIWAKI
(Gunma University)

I. Introduction.

Polya says as follows in his book[3] "What courses should the colleges offer to prospective high school teachers? We can't reasonably answer this question, unless we first answer the related question : What should the high schools offer to their students?

Our knowledge about any subject consists of information and of know-how. If you have genuine bona fide experience, of mathematical work on any level, elementary or advanced, there will be no doubt in your mind that in mathematics, know-how is much more important than mere posession of information.

To fill this gap, the teacher's curriculum should make room for creative work on an appropriate level. I attempted to give opportunity for such work by conducting seminars in problem solving".

As two main objects of mathematical education, we have

(i) How to solve mathematical problems,

(ii) How to think mathematics and plausible reasoning.

In (ii) be emphasize

(a) Induction and analogy in mathematics,

(b) Patterns of a plausible inference.

The latter (b) seems to me that it is as same as Jutsu — a technical word used by the Wasan experts.

The main object of this paper is to point out polya's way of thinking by refering to examples of one of the old Japanese manuscripts [4] and the recent study[2].

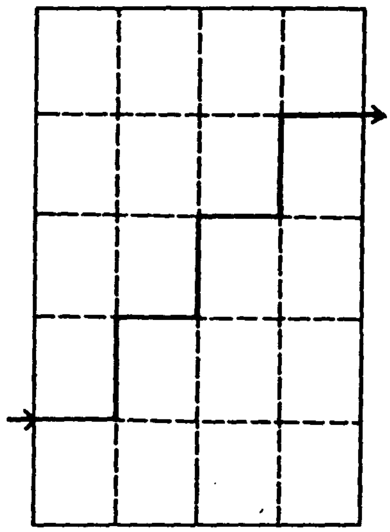
II. An example of "Kaisan-ki".

Masashige Yamada (? - ?) lived in Nara Prefecture and wrote a book "Kaisan-ki"[5] in 1659. One of the questions in his book as follows :

How do we cut down a rectangular cloth having the two sides 50, 32cm to quadrable one at a single stroke?

The solution is a square having a side 40cm.

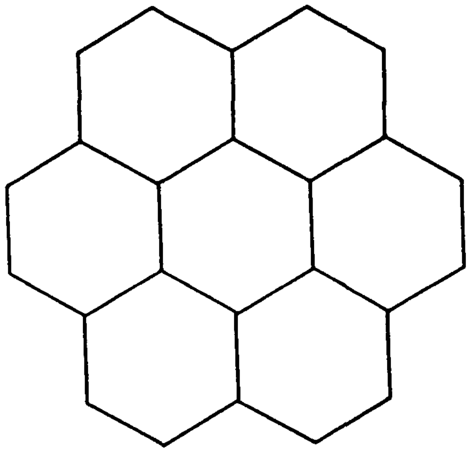
This problem is solved easily. The method of making the square is showed as a figure.



III. Problem of simple geometrical figures on pavement.[2]

About ten years ago, I went to India and attended the Celebration of the 1500th Birth Anniversary of Aryabatha I, at the invitation of the Indian National Science Academy. At that time, when I saw the paved road to the Taj Mahal, I thought of a problem.

There is a regular hexagon. Let us make six same regular hexagons around the initial one, and each of them attaches the other two and the intial one.



Cut off the above figure straight and stick on the peaces, then make up a new regular hexagon having the area of the sum of the seven ones.

How many times at least do we need to cut?

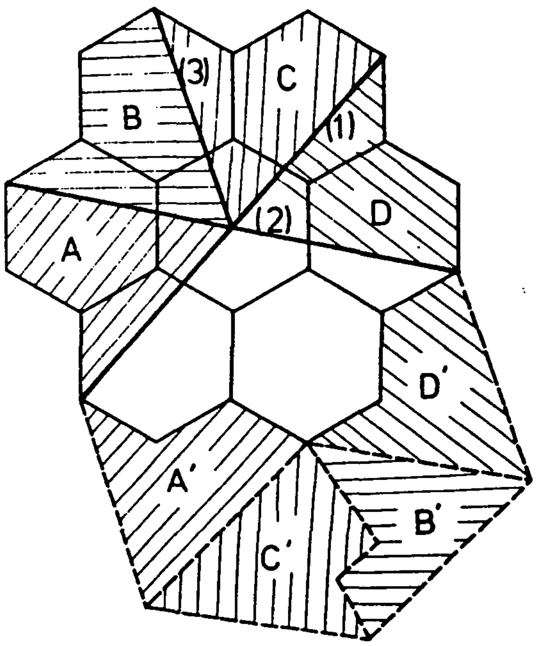
Let a be a side of the initial regular hexagon, then the area of the sum of the seven ones is

$$\frac{\sqrt{3}}{4} a^2 \times 6 \times 7 = \frac{21\sqrt{3}}{2} a^2.$$

Put x be a side of the seeking regular hexagon, then

$$\frac{\sqrt{3}}{4} x^2 \times 6 = \frac{21\sqrt{3}}{4} a^2.$$

$$\therefore x = \sqrt{7}\ a$$



The longest distance between two vertexes of a regular hexagon is

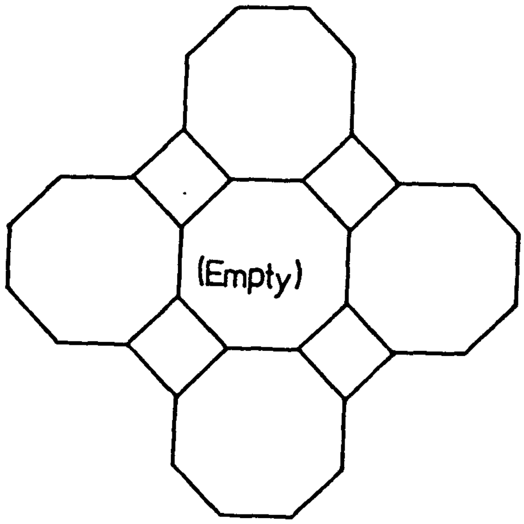
$2[a^2 + (2a)^2 - 2 \cdot 2a^2\cos 120°]^{\frac{1}{2}} = 2\sqrt{7}\ a.$

It is obvious that this passes through the center of symmetry. A half of this lenght is a side of the seeking regular hexagon.

Drawing : See the figure.

IV. Problem of geometrical figures on pavement[1]).

There is a geometrical figure consisted of four squares and four regular octagons which touch each other alternately.



Cut off the above geometrical figure straight and stick on the peaces, then make up a new square having the same area.

How many times at least do we need to cut?

Let a be a side of the regular octagon, the area is

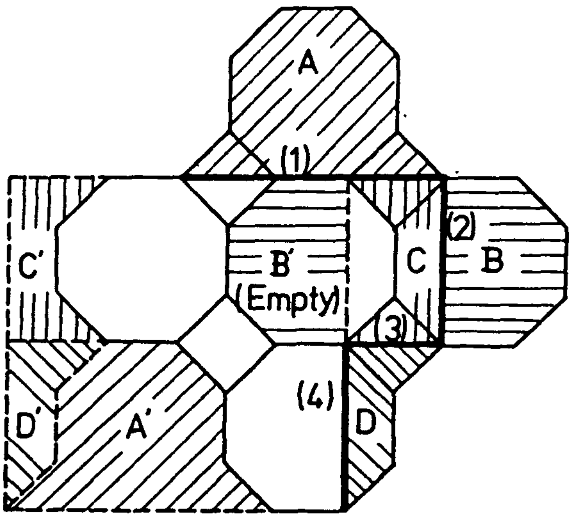
$$2(\sqrt{2} + 1)a^2.$$

Then the first geometrical figuare's area is

$$4 \times 2(\sqrt{2} + 1)a^2 + 4a^2 = \{2(\sqrt{2} + 1)a\}^2.$$

Therefore we can draw a square having a side $2(\sqrt{2} + 1)a$.

Drawing : See the figure.



V. Conclusion.

As you know the above examples, they coincide with the pattern that Prof.Polya says. The problems such as the above are most suitable examples for mathematical discovery.

In Ⅲ, for example, we change the regular hexagon to a square, then what the result is ?

In IV, we change the figure to one including the inside regular octagon, then what the result is ?

It is worthy to let students of Faculty of Education to think about how to deal with problems and how to transform them. I believe that it becomes true mathematical eductation

Supplement:

In the above conclusion, several questions are added to the lecture on ICME 5. Some problems and its solutions for the above Ⅲ and IV are shown belows.

Problem in Ⅲ. We change the regular hexagon to regular triangle or rectangular. How many times at least do we need to cut ?

Using the same notation as in the Ⅲ. Then

$$\frac{\sqrt{3}}{4} a^2 \times 6 \times 6 = \frac{\sqrt{3}}{4} x^2$$

$$\therefore x = 6a.$$

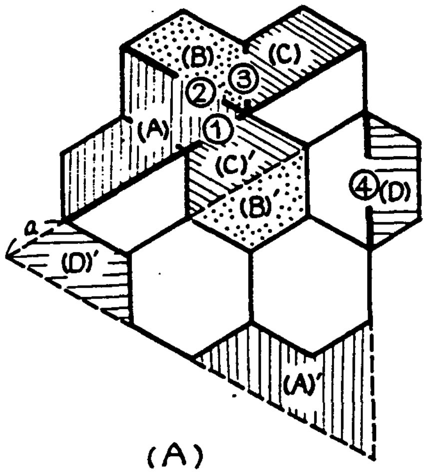
On the other hand,

$$\frac{\sqrt{3}}{4} a^2 \times 6 \times 6 = 3a \cdot 3\sqrt{3}\, a$$

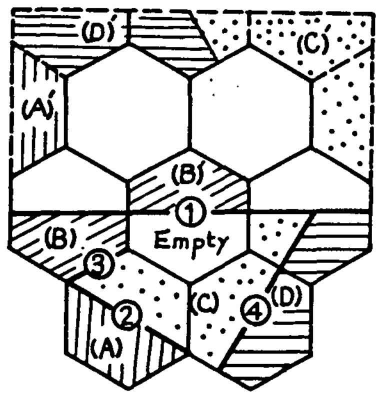
Drawing : See the figuer (A) or (B).

Remark. On the Drawing (B) is another solution. The cutting way of this is unprecedented. It owes to Mr. 王世錦(武漢建築機料工學院).

We show it below.



(A)



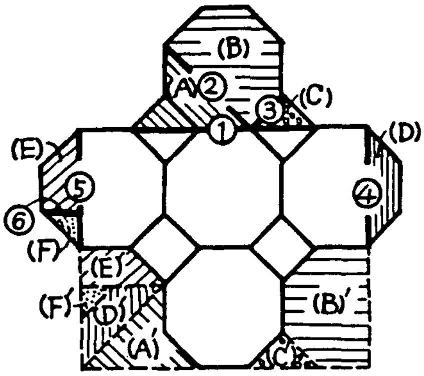
(B)

Problem in IV. We change the figure to one including the inside regular octagon. How many times at least do we need to cut?

Same notation are used in the IV. Though $x^2 = \dfrac{\sqrt{2}+1}{\sqrt{2}} a^2$, then following equality hold.

$$5(\text{regular octagon}) + 4(\text{square})$$
$$= (14 + 10\sqrt{2})a^2 = 2(\sqrt{2}+1)a \cdot (3 + 2\sqrt{2})a.$$

Then the following figure.



This paper is reading at the Fifth International Congress of Mathematical Education held on from 1984. 8. 24 to 1984. 8. 31 at Adelaide, Australia and addition some supplements.

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References

1) Yoshimasa Michiwaki : On the Mystery of Geometrical Figures, Mathematical Sciences, Puzzle II , 1977. (Japanese)

2) Yoshimasa Michiwaki : unpublished paper

3) George Polya : Mathematical Discovery, Vol.1 John Wiley & Sons. Inc. U.S.A. 1962.

4) Masashige Yamada : Kaisan - Ki, 1659. (Japanese)