A Study on the Lot-Sizing with Deterministic Dynamic Demand

決定的 動的 需要를 갖는 경우의 로트크기 決定에 관한 硏究

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MRP System에서 시스템 성능을 향상시킬 수 있는 하나의 인자는 우수한 Lot-Sizing rule의 선택과 발견이다. 널리 사용되고 있는 Lot-Sizing rule 들 중 Wagner-Whitin 알고리즘은 생산준비 비용과 재고 유지비용이 가변적일 경우 동적 로트크기 결정에 우수한 forward 알고리즘이다. 본 연구에서는 Wagner-Whitin 알고리즘의 중요 원리들을 응용하여 LDS 알고리즘이라 명명한 backward 알고리즘을 개발한다. 그리고 개발된 알고리즘을 Wagner-Whitin 알고리즘과 비교함으로 그 유효성을 테스트하고 backward 알고리즘이 가지는 계산상의 장점을 고찰한다.

1. Introduction

Material Requirements Planning (MRP) is currently being applied and used extensively in industry and is being proclaimed as the solution to many of the problems of traditional production-inventory control problems. While the latter may be true, MRP is not without its own problems, two of which are the decision concerning the appropriate lot sizing and sequencing rules to use in order to improve system performance.

In this paper the efficient lot sizing rule is forced to improve system performance. There are many lot sizing rules. Among these, Wagner - Whitin Algorithm is especially efficient to handle the dynamic lot size with various set up costs and inventory carrying costs.

But Wagner - Whitin Algorithm is forward pass algorithm. We develop backward pass algorithm, so called LDS's algorithm in this research. Forward algorithm and backward algorithm are not different in basic principles, but each has its own characteristics.

We test the efficiency of our algorithm by comparing with Wagner - Whitin algorithm.

2. Mathematical Model

As in the standard lot size formulation, we assume that the buying (or manufacturing) costs and selling price of the item are constant throughout all time periods, and consequently only the costs of inventory management are of concern.

In the t-th period, $t = 1, 2, \cdots\cdots, N$, we let

d_t : amount demanded (accured at the end of t-th period)

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i_t : interest charge per unit of inventory carried forward to period $t+1$

S_t : ordering(or set up) cost

X_t : amount ordered(or manufactured) (occured at the end of t-th period)

We assume that all period demands and costs are non-negative. The problem is to find a program $X_t \geqq 0$, $t = 1, 2, \cdots, N$, such that all demands are met at a minimum total cost.

Of course one method of solving the optimization problem is to enumerate 2^{N-1} combinations of either ordering or not ordering in each period (We assume an order is placed in the first period).

A more efficient algorithm evolves from a dynamic programming characterization of an optimal policy. Especially, Wagner-Whitin Algorithm is efficient for these problems. That algorithm is forward algorithm, ordinally. So, in this paper backward-pass algorithm is developed and differencies of forward algorithm and backward algorithm are discussed.

Let I denote the inventory entering a period and Io initial inventory :

for period t, then

$$I = I_0 + \sum_{j=1}^{t-1} X_j - \sum_{j=1}^{t-1} d_j \geqq 0 \quad \cdots\cdots (1)$$

We may write the functional equation representing the minimal cost policy for period t through N, given incoming inventory I , as

$$f_t(I) = \min_{\substack{x_t \geqq 0 \\ I+x_t \geqq d_t}} \left[i_{t-1} I + \delta(X_t) S_t + f_{t+1}(I + X_t - d_t) \right] \cdots\cdots (2)$$

Where

$$\delta(X_t) = \begin{cases} 0 & \text{if } X_t = 0 \\ 1 & \text{if } X_t > 0 \end{cases} \quad \cdots\cdots (3)$$

In period N we have

$$f_N(I) = \min_{\substack{x_N \geqq 0 \\ I+x_N = d_N}} \left[i_{N-1} I + \delta(X_N) S_N \right] \cdots\cdots (4)$$

Consequently we compute f_t, starting at $t=N$, as a function of I; ultimately we drive f_1, thereby obtaining an optimal solution as I for period 1 is specified. Theorem 2 below establishes that it is permissible to confine consideration to only $N+2-t$, $t>1$, values of I at period t.

By taking cognizance of the special properties of our model, we may formulate an alternative functional equation which has the advantage of potentially requiring less than N periods' data to an optimal program ; that is, it may be possible without any loss of optimality to narrow our program commitment to a shorter "planning horizon" than N periods on the sole basis of data for this horizon. Just as one may prove that in a linear programming model it suffices to investigate only basic sets of variables in search of an optimal solution, we shall demonstrate that in our model an optimal solution exists among a very simple class of policies.

It is necessary to postulate that $d_1 \geqq 0$ is demand in period 1 net of starting inventory. Then the fundamental proposition underlying our approach asserts that it is sufficient to consider programs in which at period t one does not both place an order and bring in inventory.

Theorem 1. There exists an optimal program such that $IX_t = 0$ for all t(where I is inventory entering period t).

Proof : Suppose an optimal program suggests both to place an order in period t and to bring in I (i.e., $IX_t > 0$).

Then it is no more costly to reschedule the purchase of I by including the purchase of I by including the quantity in X_t, for this alteration does not incur any additional ordering cost and does save the cost $i_{t-1} I \geqq 0$.

Note that the theorem does not hold if our model includes buying or production costs which are not constant and identical for all periods. In the latter case, economics of scale might very well call for the carrying of inventory into period t even when an order or set up takes place in t.

Two corollaries follow from the theorem.

Theorem 2. There exists an optimal program such that for all t $X_t = 0$ or $\sum_{j=t}^{k} d_j$ for some k, $t \leqq k \leqq N$.

Proof : Since all demands must be met, any other value for X_t implies there exists a period $t^* \geqq t$ such that $IX_{t^*} > 0$; but theorem 1 assures that it is sufficient to consider programs in which such a condition does not arise.

The implication of theorem 2 is that we can limit the values of I in (2) for period t to zero and the cumulative sum of demand for periods t up to N. If initial inventory is zero, then only $N(N+1)/2$ different values of I in toto over the entire N periods need be examined.

Theorem 3. There exists an optimal program such that if $d_{t^{**}}$ is satisfied by some X_{t^*}, $t^{**} > t^*$, then d_t, $t = t^*+1, t^*+2, \cdots, t^{**}-1$ is also satisfied by X_{t^*}.

Proof : In a program not satisfying the theorem, either I for period t^* is positive or I for period t^{**} is brought into some period t', $t^{**} > t' > t^*$, where $X_{t'} > 0$; but again by theorem 1, it is sufficient to consider programs in which such conditions do not arise.

We next investigate a condition under which we may divide our problem into two smaller subproblems.

Theorem 4. Given that $I = 0$ for period t, it is optimal to consider periods 1 through $t-1$ by themselves. i.e., it is optimal to consider periods t through N by themselves.

Proof :by hypothesis, (2) in period $t-1$ for the N period model is

$$f_{t-1}(I) = \min_{\substack{x_{t-1} \geqq 0 \\ I + x_{t-1} = d_{t-1}}} \left[i_{t-2} I + \delta(X_{t-1}) S_{t-1} + f_t(0) \right] \cdots\cdots (5)$$

and for the $t-1$ period model is correspondingly

$$g_{t-1}(I) = \min_{\substack{x_{t-1} \geqq 0 \\ I + x_{t-1} = d_{t-1}}} \left[i_{t-2} I + \delta(X_{t-1}) S_{t-1} \right] \cdots\cdots (6)$$

But the functional relations (5) and (6) differ only by a constant $f_t(0)$. Consequently what is optimal for (6) is optimal for (5), and by the recursive structure of the model, the latter conclusion continues to hold for all the earlier periods.

We may now offer an alternative formulation to (2). Let F(t) denote the minimal cost program for periods t through N. Then

$$F(t) = \min \begin{cases} \min_{t < j \leqq N} \left[S_t + \sum_{h=t}^{j-1} \sum_{k=h+1}^{j} i_h d_k + F(j+1) \right] \\ S_t + F(t+1) \end{cases} \cdots\cdots (7)$$

Where

$F(t) = S_N$

$F(N+1) = F(\cdot) = 0$

That is, the minimum cost for the first $N-t+1$ periods comprises a set up cost in period t, plus charges for filling demand d_k, $k = t+1, \cdots, j$, by carrying inventory from period t, plus the cost of adopting an optimal policy in period $N+1$ through N taken by themselves. Theorem 2, 3, and 4 guarantee that at period t we shall find an optimum program of this type.

When the present formulation, (7) is computed, starting at $t=N$. At any period t, (7) implies that only $N-t+1$ policies need to be considered.

The minimum in (7) need not be unique, so that there may be alternative optimal solutions. When we derive F(1), we shall have solved the problem for 1 is the last period to be considered.

Finally we come to what is perhaps the most interesting property of our model.

The Planning Horizon Theorem.

If at periods t^{**} the minimum in (7) occurs for $j = t^{**} \geqq t^*$, then in periods $t < t^*$ it is sufficient to consider only $t \leqq j \leqq t^{**}$. In particular, if $t^* = t^{**}$, then it is sufficient to consider programs such that $X_t{}^{**} > 0$.

Proof : Without loss of optimality we restrict our attention to programs of the form specified in theorems 1 - 4. Suppose a program suggests that $d_t{}^{***}$ is satisfied by X_t, where $t^{***} > t^{**} > t^* > t$. Then by theorem 3 $d_t{}^*$ is also satisfied by X_t. But by hypothesis we know that costs are not increased by rescheduling the program to let $d_t{}^{**}$ be satisfied by $X_t{}^* > 0$.

The planning horizon theorem states in part that if it is optimal to incur a set up cost in period t^* when period t^* through N are considered by themselves, then we may let $X_t{}^* > 0$ in the N period model without foregoing optimality. By theorems 1 and 4 it follows further that we may adopt an optimal program for periods t^* through N considered separately.

3. the Algorithm

The algorithm at period t^*, $t^* = N$, $N-1$, $N-2 \cdots\cdots 2, 1$, may be generally stated as

1. Consider the policy of ordering at period t^*, and filling demands d_t, $t = t^*$, $t^*+1, \cdots\cdots$, t^{**}, by this order.

2. Determine the total cost these $(N-t^*+1)$ different policies by adding the ordering and holding cost associated with placing the order at period t^*, and the cost of acting optimally for periods t^*+1 through N considered by themselves.

3. From these $(N-t^*+1)$ alternatives, select the minimum cost policy for period t^* through N considered independently.

4. Proceed to period t^*-1 (or stop if $t^*=1$).

Table 1 portrays the symbolic scheme for the algorithm. The notation $(t^*, t^*+1, \cdots\cdots, t^{**})$ $t^{**}+1$, $t^{**}+2, \cdots\cdots$, N in Table 1 indicates that an order is placed in period t^* to cover the demands of d_t, $t = t^*, t^*+1, t^*+2, \cdots\cdots, t^{**}$, and the optimal policy is adopted for periods $t^{**}+1$, $t^{**}+2, \cdots\cdots$, N considered separately. At the bottom of the table we record the minimum cost plan for periods t^* through N.

In general, it may be necessary to test N policies at the first period, implying a table of $N(N+1)/2$ entries (versus 2^{N-1} for all possibilities).

As we shall see, the number of entries usually is much smaller than this number if we make full use of the planning horizon theorem.

Table 1. Computation Procedure of LDS Algorithm

Month	N	N − 1	N − 2	N − 3	…	1
Ordering cost	S_N	S_{N-1}	S_{N-2}	S_{N-3}	…	S_1
Demand	d_N	d_{N-1}	d_{N-2}	d_{N-3}	…	d_1
(t) t + 1, t + 2, …, N	(N)	(N − 1) N	(N−2), N−1, N	(N − 3), N−2, N−1, N	…	(1) 2, 3, ……N
(t, t + 1) t + 2, t + 3 …N		(N − 1, N)	(N−2, N−1) N	(N−3, N−2) N−1, N	…	(1, 2) 3, 4…N
(t, t+1, t+2) t+3, t +4 …N			(N−2, N−1, N)	(N−3, N−2, N−1), N	…	(1, 2, 3) 4, 5 …… N
(t, t+1, t+2, t +3) t + 4 …N				(N−3, N−2, N−1, N)	…	(1, 2, 3, 4) 5, 6 …… N
						⋮
Minimum Cost Optimal policy (t, t +1, ……, N)	(N)	(N−1, N)	(N−2, N−1, N)	(N−3, N−2, N−1, N)	…	(1, 2, 3, ……, N)

4. LDS's backward Model and Wagner-Whitin's forward Model.

Wagner-Whitin's algorithm is a efficient forward algorithm for a solution to the following dynamic version of the economic lot size model is given : allowing the possibility of demands for a single item, inventory holding charges, and set up costs to vary over N periods. We desire a minimum total cost inventory management scheme which satisfies know demand in every period.

Our backward algorithm is not different from Wagner-Whitin's forward algorithm in fundamental principles. But computation procedures are different each other.

Two algorithm are summerized as follows :

a. Wagner-Whitin's forward Model.

F(t) ; the minimal cost program for period 1 through t

$$F(t) = \min \begin{cases} \min\limits_{1 \le j < t} \left[S_j + \sum\limits_{h=j}^{t-1} \sum\limits_{k=h+1}^{t} i_n d_k + F(j-1) \right] \\ S_t + F(t-1) \end{cases}$$

Where

$$F(1) = S_1$$
$$F(0) = 0$$

computed, starting at $t = 1$

b. LDS's backward Model

F(t) ; the minimal cost program for period to t through N

$$F(t) = \begin{cases} \min \min\limits_{t < j \le N} \left[S_t + \sum\limits_{h=t}^{j-1} \sum\limits_{k=h+1}^{j} i_n d_k + F(j+1) \right] \\ S_t + F(t+1) \end{cases}$$

Where

$$F(N) = S_N$$
$$F(N+1) = F(\cdot) = 0$$

computed, starting at $t = N$.

5. An Example

Table 2 presents a sample set of data for a 12 month period; Table 3 contains the specific calculation and compares LDS's Backward Model with Wagner - Whitin's for ward Model.

Table 2 Data for a 12 month period

month (t)	d_t	s_t	i_t
1	69	85	1
2	29	102	1
3	36	102	1
4	61	101	1
5	61	98	1
6	26	114	1
7	34	105	1
8	67	86	1
9	45	119	1
10	67	110	1
11	79	98	1
12	56	114	1
Average	52.5	102.8	1

Table 3 Wagner - Whitin's forward algorithm and LDS's backward algorithm.

Month t	1	2	3	4	5	6	7	8	9	10	11	12	
Ordering cost	85	102	102	101	98	114	105	86	119	110	98	114	
Demand	69	29	36	61	61	26	34	67	45	67	79	56	
	85	187	216	287	375	462	505	555	674	710	808	903	
		114	223	277	348	401	496	572	600	741	789	864	
			186	345	399	400	469	630		734		901	1
				369			502	670					
Minimum cost	85	114	186	277	348	400	469	555	600	710	789	864	
Optimal policy *	1	1, 2	1, 2, 3	3, 4	4, 5	4, 5, 6	5, 6, 7	8	8, 9	10	10, 11	11, 12	
	911	852	790	688	641	614	500	426	383	264	212	114	
	864	826	750	705	624	543	512	395	340	303	154		
	874	847		714	587		526	419		301			
	956			711									2
Minimum cost	864	826	750	688	587	543	500	395	340	264	154	114	
Optimal policy**	(1, 2)	(2, 3)	(3, 4)	(4)	(5, 6, 7	(6, 7)	(7)	(8, 9)	(9, 10)	(10)	(11, 12)	(12)	

1. Wagner-Whitin's forward algorithm
2. LDS's backward algorithm

Only the last period is shown ;

* 5, 6, 7 indicates that the optimal policy for periods 1 through 7 is to order in period 5 to satisfy d_5, d_6, and d_7, and adopt an optimal policy for periods 1 through 4 considered separately.

** (5, 6, 7) indicates that the optimal policy for periods 5 through 12 is to order in period 5 to satisfy d_5, d_6, and d_7, and adopt an optimal policy for period 8 through 12 considered separately.

6. Conclusion

One of factors which are able to improve the system performance is selection and development of the efficient lot sizing rules in MRP. Among many lot sizing rules using widely, Wagner - Whitin algorithm is specially efficient to handle the dynamic lot size with various set up costs and inventory carrying costs.

Our algorithm developed by revising the important principles of Wagner - Whitin algorithm, is backward pass algorithm. So our algorithm and Wagner - Whitin algorithm is not different in the basic principles. But forward algorithm and backward algorithm have its own characteristics, respectively. In computation structure, backward algorithm is convenient for changing initial periods in contrast with forward algorithm being convenient for changing last periods of planning borizon. In our example our backward algorithm is more efficient than Wagner - Whitin forward algorithm in computation.

This phenominon is not general and appears differently case by case. But this phenominon indicates that our backward pass algorithm is not always more efficient than Wagner - Whitin forward pass algorithm, however, at least in more efficient than foreward algorithm in some cases. Therefore, our algorithm is valuable sufficiently. Especially, MRP system has possibility of changing the demand in initial periods before performing some planning. In this case, backward algorithm will be more useful.

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