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Cyclic Factorial Association Scheme Partially Balanced Incomplete Block Designs

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ABSTRACT

Cyclic Factorial Association Scheme (CFAS) for incomplete block designs in a factorial experiment is defined. It is a generalization of EGD/(2^n-1)-PBIB designs defined by Hinkelmann (1964) or Binary Number Association Scheme (BNAS) named by Paik and Federer (1973). A property of PBIB designs having CFAS is investigated and it is shown that the structural matrix NN' of such designs has a pattern of multi-nested block circulant matrix. The generalized inverse of $(rI-NN'/k)$ is obtained. Generalized Cyclic incomplete block designs for factorial experiments introduced by John (1973) are presented as the examples of CFAS-PBIB designs. Finally, the relationship between CFAS and BNAS in block designs is briefly di scussed.

1. Introduction

In a factorial expertment with n factors $F_1,\ F_2,\dots,F_n$ at $m_1,\ m_2,\dots,m_n$ levels respectively, consider the following association scheme: The two treatment combinations $(i_1, i_2,\dots,i_n)$ and $(j_1,j_2,\dots,j_n)$ are $(p_1,p_2,\dots,p_n)$ th associates, where $p_k=0,1,\dots,m_k-1$ for $k=1,2,\dots,n$, if

$$(i_1,i_2,\dots,i_n)+(p_1,p_2,\dots,p_n)=(j_1,j_2,\dots,j_n), \tag{1.1}$$

where $i_k+p_k=j_k \bmod m_k$ $(k=1,2,\dots,n)$. This association could be called a "Cyclic Factorial Association Scheme (CFAS)." The relationship in (1.1) implies that two treatment combinations $(i_1,i_2,\dots,i_n)$ and $(i_1-p_1,\ i_2-p_2,\dots,i_n-p_n)$ are also $(p_1,p_2,\dots,p_n)$ th associates, because $(i_1-p_1,\ i_2-p_2,\dots,i_n-p_n)+(p_1,p_2,\dots,p_n)=(i_1,i_2,\dots,i_n)$. In a block design, $\lambda(p_1,p_2,\dots,p_n)$ will denote the number of times these treatments which are $(p_1,$

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$p_2, \ldots, p_n$)th associates each other occur together in a block.

Cyclic Factorial Association Scheme is a generalization of EGD/(2^{n-1})-PBIB designs (EGD means extended group divisible) defined by Hinkelmann (1964) or Binary Number Association Scheme (BNAS) named by Paik and Federer (1973). The relationship between CFAS and BNAS in incomplete block designs is discussed in Section 6.

We will investigate a property of Partially Balanced Incomplete Block (PBIB) designs having CFAS and the structural property of the designs which is related to the block incidence matrix N of the designs will be termed Property C. In a PBIB design, it can be shown that if the design has the CFAS the structural matrix NN' has a pattern of multi-nested block circulant. In Section 4, we will discuss about the inverse of the multi-nested block circulant matrix. Generaltzed Cyclic incomplete block designs for factorial experiments introduced by John (1973) are discussed as examples of CFAS-PBIB designs in Section 5.

2. Preliminaries

In the factorial experiment considered in the previous section, the number of treatment combinations is $v=\prod_{i=1}^{n} m_i$. Let the ith treatment combination be denoted by the n-tuple $(i_1, i_2, \ldots, i_n)$, where $i_s=0, 1, \ldots m_s-1$ for all $s=1, 2, \ldots, n$. Treatment combinations are written in lexicographical order, i.e., the subscript corresponding to the last factor is changed first, and are isomorphic with treatments $0, 1, \ldots, v-1$. The relationship between the order of treatment i and the corresponding treatment combination can be expressed by

$$i=\left[\sum_{s=1}^{n-1}\left(\prod_{k=s+1}^{n} m_k\right)i_s\right]+i_n. \tag{2.1}$$

Let the v treatment combinations be allocated to b blocks each of k plots with the ith treatment replicated r times. The usual intra-block model will be assumed, namely

$$y_{ij}=\mu+\tau_i+\beta_j+\epsilon_{ij} \quad (i=0, 1, \ldots, v-1;\ j=1, 2, \ldots, b), \tag{2.2}$$

where y_{ij} is the yield of the plot in the jth block to which ith treatment has been applied, μ is a general mean, τ_i is the ith treatment effect, β_j the jth block effect and the ϵ_{ij} are independent normal variates with zero means and homogeneous variances σ^2. Let T_i be the total yield of all plots receiving the ith treatment and B_j the total yield of all plots in the jth block. The incidence matrix N is defined to have a row for each treatment and a column for each block and the elements represent the number of times each treament occurs in a block.

The reduced normal equations for the treatment effects in the intra-block model eliminating blocks, are

$$A\hat{\boldsymbol{\tau}} = \boldsymbol{Q}, \tag{2.3}$$

where

$$A = rI - (1/k)NN'$$
$$\boldsymbol{Q} = \boldsymbol{T} - (1/k)N\boldsymbol{B} \tag{2.4}$$

and where $\hat{\boldsymbol{\tau}}$ is the intra-block estimates of $\boldsymbol{\tau} = (\tau_0, \tau_1, \cdots, \tau_{v-1})'$. Now, suppose that the block design is a PBIB design having CFAS in a $m_1 \times m_2 \times \cdots \times m_n$-factorial experiment. In this case, by definition, (1) with reference to any specified treatment, the remaining $v-1$ fall into m(say) sets, the $(p_1, p_2, \cdots, p_n)$th set of which occurs with the specified treatment in $\lambda(p_1, p_2, \cdots, p_n)$ blocks and contains $n(p_1, p_2, \cdots, p_n)$ treatments, the numbers $\lambda(p_1, p_2, \cdots, p_n)$ and $n(p_1, p_2, \cdots, p_n)$ being the same, respectively, regardless of the treatment specified, (2) if we call the treatments that lie in a block $\lambda(p_1, p_2, \cdots, p_n)$ times with a specified tratment θ, the $(p_1, p_2, \cdots, p_n)$th associates of θ, the number of treatments common to the $(p_1, p_2, \cdots, p_n)$th associates of θ and $(q_1, q_2, \cdots, q_n)$th associates of ϕ, where θ and ϕ are the $(k_1, k_2, \cdots, k_n)$th associates, is $p(k_1, k_2 \cdots, k_n)/(p_1, p_2, \cdots, p_u)$, $(q_1, q_2, \cdots, q_n)$, this number being the same for any pair of $(k_1, k_2, \cdots, k_n)$th associates.

Note: If two treatment combinations $(i_1, i_2, \cdots, i_n)$ and $(j_1, j_2, \cdots, j_n)$ are $(p_1, p_2, \cdots, p_n)$th associates in a PBIB design the following should also hold:

$$(j_1, j_2, \cdots, j_n) + (p_1, p_2, \cdots, p_n) = (i_1, i_2, \cdots, i_n), \tag{2.5}$$

where $j_k + p_k = i_k \bmod m_k$ $(k=1, 2, \cdots, n)$. So, from (1.1) and (2.5)

$$\lambda(i_1 - j_1, i_2 - j_2, \cdots, i_n - j_n) = \lambda(j_1 - i_1, j_2 - i_2, \cdots, j_n - i_n). \tag{2.6}$$

3. Property of Cyclic Factorial Association Scheme PBIB Designs

Consider a CFAS-PBIB design in a factorial experiment and let $NN' = [\lambda_{ij}]$, $i = 0, 1, \cdots v-1$; $j = 0, 1, \cdots, v-1$, then it is well known that if the ith treatment and jth treatment are $(p_1, p, \ldots, p_n)$th associates $\lambda_{ij} = \lambda(p_1, p_2, \cdots, p_n)$. In an $m_1 \times m_2$-factorial experiment, the square matrix NN' can be partitioned as

$$NN' = [M(i_1, j_1)], \quad i_1, j_1 = 0, 1, \cdots, m_1 - 1, \tag{3.1}$$

where $M(i_1, j_1)$ are $m_2\times m_2$ square matrix. Let $M(i_1, j_1)=[M_{i_1j_1}(i_2, j_2)]$, $i_2, j_2=0, 1, \cdots, m_2-1$, and suppose $i_1+p_1=j_1$ mod m_1, and $i_2+p_2=j_2$ mod m_2, for $i_2, j_2=0, 1, \cdots, m_2-1$, then $M_{i_1j_1}(i_2, j_2)=\lambda(p_1, p_2)$, where p_1 is a constant but p_2 varies according to the values of i_2 and j_2. If the design is PBIB with CFAS the $M(i_1, j_1)$ becomes a circulant matrix. That is

$$M(i_1, j_1)=\begin{bmatrix} \lambda(p_1, 0) & \lambda(p_1, 1)\cdots & \lambda(p_1, m_2-1) \\ \lambda(p_1, m_2-1) & \lambda(p_1, 0)\cdots & \lambda(p_1, m_2-2) \\ & \cdots\cdots\cdots\cdots & \\ \lambda(p_1, 1) & \lambda(p_1, 2) & \lambda(p_1, 0) \end{bmatrix},$$

Furthermore $M(i_1+1, j_1+1)=M(i_1, j_1)$ mod m_1 for i_1+1 and j_1+1, since if $i_1+p_1=j_1$ then $i_1+1+p_1=j_1+1$ mod m_1.

Therefore, we can say that matrix NN' is a doubly (or second order) nested block circulant matrix.

The extension to the multi-factor experiments is straightforward. In an $m_1\times m_2\times\cdots\times m_n$-factorial experiment with PBIB design having CFAS, the structural matrix NN' can be partitioned as follows

$$\begin{aligned} &NN'=[M(i_1, j_1)],\quad i_1, j_1=0, 1, \cdots, m_1-1, \\ &M(i_1, j_1)=[M_{i_1j_1}(i_2, j_2)],\quad i_2, j_2=0. 1, \cdots, m_2-1, \qquad (3.2) \\ &\qquad\cdots\cdots\cdots\cdots \\ &M_{i_1i_2\cdots i_{n-2},\, j_1j_2\cdots j_{n-2}}(i_{n-1}, j_{n-1})=[M_{i_1i_2\cdots i_{n-1},\, j_1j_2\cdots j_{n-1}}(i_n, j_n)], \\ &\qquad\qquad i_n, j_n=0, 1, \cdots, m_n-1, \end{aligned}$$

$$M_{i_1i_2\cdots i_{n-1},\, j_1j_2\cdots j_{n-1}}(i_n, j_n)=\begin{bmatrix} \lambda(p, 0) & \lambda(p, 1)\cdots & \lambda(p, m_n-1) \\ \lambda(p, m_n-1) & \lambda(p, 0)\cdots & \lambda(p, m_n-2) \\ & \cdots\cdots\cdots\cdots & \\ \lambda(p, 2) & \lambda(p, 3)\cdots & \lambda(p, 1) \\ \lambda(p, 1) & \lambda(p, 2)\cdots & \lambda(p, 0) \end{bmatrix},$$

where $p=(p_1, p_2, \cdots, p_{n-1})$, that is $(i_1, i_2, \cdots, i_{n-1})+(p_1, p_2, \cdots, p_{n-1})=(j_1, j_2, \cdots, j_{n-1})$, where $i_k+p_k=j_k$ mod m_k $(k=1, 2, \cdots, n-1)$. Clearly $M_{i_1i_2\cdots i_{n-1},\, j_1j_2\cdots j_{n-1}}(i_n, j_n)$ is a circulant matrix. Also, since $(i_1, i_2, \cdots, i_{n-1}+1)+(p_1, p_2, \cdots, p_{n-1})=(j_1, j_2, \cdots, j_{n-1}+1)$,

$$\begin{aligned} &M_{i_1i_2\cdots(i_{n-1}+1),\, j_1j_2\cdots(j_{n-1}+1)}(i_n, j_n) \\ &\qquad =M_{i_1i_2\cdots i_{n-1},\, j_1j_2\cdots j_{n-1}}(i_n, j_n) \text{ mod } m_{n-1} \text{ for } i_{n-1}+1 \text{ and } j_{n-1}+1. \end{aligned}$$

Therefore, $M_{i_1i_2\cdots i_{n-2},\, j_1j_2\cdots j_{n-2}}(i_{n-1}, j_{n-1})$ is a second order nested block circulant matrix. Extending this argument to the vxv matrix NN' in (3.2), we may understand that the matrix NN' is an nth order nested block circulant matrix.

Let R_n^i is defined to be an nxn circulant matrix whose first row has unity in the ith column and zero elsewhere, where $i=0, 1, \cdots, n-1$, then the circulant matrix

$$M=\{m_0, m_1, \cdots, m_{n-1}\}=\begin{bmatrix} m_0 & m_1 \cdots m_{n-2} & m_{n-1} \\ m_{n-1} & m_0 \cdots m_{n-3} & m_{n-2} \\ & \cdots\cdots\cdots & \\ m_1 & m_2 \cdots m_{n-1} & m_0 \end{bmatrix}$$

can be written as

$$M=\sum_{i=0}^{n-1} m_1 R_n^i.$$

Let $M_i=\{m_{i0}, m_{i1}, \cdots, m_{i,n-1}\}$ and $M=\{M_0, M_1, \cdots, M_{m-1}\}$, i.e., M is an mnxmn second order nested block circulant, then

$$M_i=\sum_{j=0}^{n-1} m_{ij} R_n^j, \quad M=\sum_{i=0}^{m-1} R_m^i \otimes M_i,$$

where $\otimes$ denotes the right Kronecker product of matrices, so

$$M=\sum_{i=0}^{m-1} R_m^i \otimes \left(\sum_{j=0}^{n-1} m_{ij} R_n^j\right)$$

$$=\sum_{i=0}^{m-1} \sum_{j=0}^{n-1} m_{ij} R_m^i \otimes R_n^j.$$

Extending the same process to the nth order nested block circulant matrix NN' in (3.2), it can be written as

$$NN'=\sum_{i_1=0}^{m_1-1} \sum_{i_2=0}^{m_2-1} \cdots \sum_{i_n=0}^{m_n-1} h(i_1, i_2, \cdots, i_n) \left(\prod_{j=1}^{n} \otimes R_{m_j}^{i_j}\right), \tag{3.3}$$

where $h(i_1, i_2, \cdots, i_n)$ are constants which correspond to the first row elements of NN' and $\prod_{j=1}^{n} \otimes R_{m_j}^{i_j}=R_{m_1}^{i_1} \otimes R_{m_2}^{i_2} \otimes \cdots \otimes R_{m_n}^{i_n}$.

The structural property of matrix N or NN' which is described in (3.3) will be termed Property C.

4. Inverse of Multi-Nested Block Circulant Matrix

A solution to the normal equation (2.3), for connected designs, is given by

$$\hat{\tau}=\Omega \boldsymbol{Q}, \tag{4.1}$$

where Ω is any generalized inverse of A, that is, satisfies $A\Omega A=A$. We can take $\Omega^{-1}=A+J$, where J is the vxv matrix with all elements unity, and it can be shown that $\Omega-(1/v^2)J$ is the Moore-Penrose's generalized inverse of A. In a CFAS-PBIB design, the matrix NN' has the structural Property C, so the matrix $(A+J)$ also has the Property C. Let

$$A+J=\sum_{i_1=0}^{m_1-1}\sum_{i_2=0}^{m_2-2}\cdots\sum_{i_n=0}^{m_n-1}a(i_1,i_2,\cdots,i_n)\left(\prod_{j=1}^{n}\otimes R_{m_j}^{i_j}\right). \tag{4.2}$$

We will prove that the matrix Ω also has the following form

$$\Omega=\sum_{i_1=0}^{m_1-1}\sum_{i_2=0}^{m_2-1}\cdots\sum_{i_n=0}^{m_n-1}c(i_1,i_2,\cdots,i_n)\left(\prod_{j=1}^{n}\otimes R_{m_j}^{i_j}\right) \tag{4.3}$$

for some constants $c(i_1,i_2,\cdots,i_n)$.

Suppose $(A+J)\Omega=I$, i.e.,

$$\left[\sum_{i_1}\sum_{i_2}\cdots\sum_{i_n}a(i_1,i_2,\cdots,i_n)\left(\prod_{j=1}^{n}\otimes R_{m_j}^{i_j}\right)\right]\left[\sum_{i_1}\sum_{i_2}\cdots\sum_{i_n}c(i_1,i_2,\cdots,i_n)\left(\prod_{j=1}^{n}\otimes R_{m_j}^{i_j}\right)\right]=I, \tag{4.4}$$

then, since $R_{m_j}^0$, $R_{m_j}^1,\cdots,R_{m_j}^{m_j-1}$ are linearly independent for $j=1,2,\cdots,n$, from (4.4), we obtain

$$(A+J)'\boldsymbol{c}=\boldsymbol{d}, \tag{4.5}$$

where $\boldsymbol{c}=(c(0,0,\cdots,0),\ c(0,0,\cdots,1),\cdots,\ c(m_1-1,\ m_2-1,\cdots,m_n-1))'$ and $\boldsymbol{d}=(1,0,\cdots,0)'$. We know that the matrix $(A+J)$ is a symmetric and nonsingular, so

$$\boldsymbol{c}=(A+J)^{-1}\boldsymbol{d}. \tag{4.6}$$

This means that there exists Ω such that it has the same structural pattern as $(A+J)$, namely, matrix Ω also has the Property C.

Cotter et al. (1973) have shown that if Ω can be written as (4.3), then the resulting design has orthogonal factorial structure. An incomplete block design is said to have orthogonal factorial structure if the adjusted treatment sum of squares from the intra-block model can be partitioned orthogonally into the main effect and interaction sums of squares.

Now, we will show the solution to the equation (4.5). If the matrix $A+J$ is simple circulant and nonsingular the inverse is obtained as follows.

Let w_0, $w_1,\cdots,w_{v-1}$ be the distinct roots of $z^v=1$, where $w_0=1$, $w_j=\cos\frac{2j\pi}{v}+i\sin\frac{2j\pi}{v}$ for $j=1,2,\cdots,v-1$, then the eigenvalues of the circulant matrix $A+J=\{a_0,a_1,\cdots,a_{v-1}\}$ are

$$\theta_j=a_0+a_1w_j+a_2w_i^2+\cdots+a_{v-1}w_j^{v-1},\quad j=0,1,\cdots,v-1,$$

and the corresponding right eigenvectors are

$$\boldsymbol{x}_j=(1,w_j,w_j^2,\cdots,w_j^{v-1})',\quad j=0,1,\cdots,v-1.$$

Let $X=(\boldsymbol{x}_0,\boldsymbol{x}_1,\cdots,\boldsymbol{x}_{v-1})$, then, since the first row of X is $\mathbf{1}'$, where $\mathbf{1}$ is $v$xl column vector with all elements unity, the first column of the inverse matrix of X should be $(1/v)\mathbf{1}$ and the jth column of X^{-1} is

$$(1/v)(1, w_1^{v-j}, w_2^{v-j}, \cdots, w_{v-1}^{v-j})' = (1/v)(1, \bar{w}_1^j, \bar{w}_2^j, \cdots \bar{w}_j^{v-1})'$$

for $j=0,1,\cdots,v-1$, where $\bar{w}_h^j$ is the conjugate of w_h^j.

Since $X^{-1}(A+J)X=\text{diag}(\theta_0, \theta_1, \cdots, \theta_{v-1})$, $(A+J)^{-1}=X\ \text{diag}(\theta_0, \theta_1, \cdots, \theta_{v-1})^{-1}X^{-1}$, and therefore the first row of $(A+J)^{-1}$ is

$$(c(0), c(1), \cdots, c(v-1)) = (1/v)\mathbf{1}'[\text{diag}(\theta_0, \theta_1, \ldots, \theta_{v-1})]^{-1}X^{-1},$$

that is

$$c(j) = \frac{1}{v}\sum_{h=0}^{n-1}\frac{\bar{w}_h^j}{\theta_h}, \quad j=0,1,\cdots,v-1.$$

The extension to the nth order nested block circulant matrix is not difficult. Let $w_0(i)=1$, $w_1(i), \cdots, w_{m_i-1}(i)$ be the distinct m_i roots of $z^{m_i}=1$ for $i=1,2,\cdots,n$, and let

$$\boldsymbol{x}_j^{(i)} = (1, w_j(i), \ w_j^2(i), \cdots, w_j^{m_i-1}(i))',$$

then it can be shown that the eigenvalues of $(A+J)$ in (4.2) are

$$\theta(i_1, j_2, \cdots, j_n) = (\text{the first row of } (A+J))\Big(\prod_{k=1}^{n}\otimes \boldsymbol{x}_{j_k}(k)\Big)$$

$$= \sum_{i_1}\sum_{i_2}\cdots\sum_{i_n} a(i_1, i_2, \cdots, i_n)\Big(\prod_{k=1}^{n} w_{j_k}^{i_k}(k)\Big), \tag{4.7}$$

$$j_k=0,1,\cdots,m_k-1 \text{ for } k=1,2,\cdots,n.$$

Let $X=\prod_{i=1}^{n}\otimes X_i$, where $X_i=(\boldsymbol{x}_0(i), \ \boldsymbol{x}_1(i), \cdots, \boldsymbol{x}_{m_i-1}(i))$, then

$$(A+J)^{-1} = X[\text{diag}(\theta(0,0,\ldots,0), \theta(0,0,\cdots,1), \cdots,$$
$$\theta(m_1-1, m_2-1, \cdots, m_n-1))]^{-1}X^{-1}. \tag{4.8}$$

Therefore, the $(j_1, j_2, \cdots, j_n)$th element of the first row of $(A+J)^{-1}$ is

$$c(j_1, j_2, \cdots, j_n) = \Big(\prod_{i=1}^{n} m_i\Big)^{-1}\Big[\sum_{i_1}\sum_{i_2}\cdots\sum_{i_n}\theta^{-1}(i_1, i_2, \cdots, i_n)\Big(\prod_{k=1}^{n}\bar{w}_{j_k}^{i_k}(k)\Big)\Big]. \tag{4.9}$$

We understand that the matrix $(A+J)$ is symmetric. Utilizing the fact that eigen values of a symmetric matrix are real, so that the imaginary parts in (4.7) and (4.9) must be identically zero, we have, therefore,

$$c(j_1, j_2, \cdots, j_n) = \Big(\prod_{i=1}^{n} m_i\Big)^{-1}\sum_{i_1}\sum_{i_2}\cdots\sum_{i_n}\theta^{-1}(i_1, i_2, \cdots, i_n)\Big(\prod_{k=1}^{n}\cos(2i_k j_k\pi/m_k)\Big), \tag{4.10}$$

where

$$\theta(j_1, j_2, \cdots, j_n) = \sum_{i_1}\sum_{i_2}\cdots\sum_{i_n} a(i_1, i_2, \cdots, i_n)\Big(\prod_{k=1}^{n}\cos(2i_k j_k\pi/m_k)\Big). \tag{4.11}$$

5. Examples of CFAS-PBIB Designs in Factorial Experiments

A Generalized Cyclic set in factorial treatments is generated from an initial block consisting of k treatments (John, 1973). The jth block of the set is given by adding the jth treatment combination to each treatment combination in the initial block, where addition is defined as $(i_1, i_2, \cdots, i_n)+(j_1, j_2, \cdots, j_n)=(k_1, k_2, \cdots, k_n)$, where $i_p+j_p=k_p$ mod m_p $(p=1, 2, \cdots, n)$. Under this definition a set will consist of $v=m_1m_2\cdots m_n$ blocks. However, some sets will have a fraction, $1/q$ say, of the v blocks replicated q times. If the treatment combinations in a block are arranged in order of magnitude and they are then regarded as an nk digit number, the initial block will be the block of lowest numerical value.

Example 5.1 Consider the construction ofsets for $v=12$, $m_1=4$, $m_2=3$, $k=4$. One possible full set is generated from the block (00, 10, 21, 32). The 12 blocks are

(00, 10, 21, 32), (01, 11, 22, 30), (02, 12, 20, 31)
(10, 20, 31, 02), (11, 21, 32, 00), (12, 22, 30, 01)
(20, 30, 01, 12), (21, 31, 02, 10), (22, 32, 00, 11)
(30, 00, 11, 22), (31, 01, 12, 20), (32, 02, 10, 21).

It can be seen that suceessive blocks in any column are generated from the first block of that column by cycling the first digit of the 2 tuple i_1i_2 under reduction m_1 where necessary. Similary for rows the second digit is used, mod m_2. This particular design is also resolvable, in the sense that each treatment combination occurs once in each row. It is clear from this that if $k=m_i$, for some i, a resolvable design of v blocks in k groups of v/k blocks can be constructed.

For Genenalized Cyclic designs, it can be seen that the structural matrix NN' is multi-nested block circulant. Therefore, a Generalized Cyclic block design in a factorial experiment has a CFAS or Property C, so it has orthogonal factorial structure. In the example 5.1, $NN'=\{M_0, M_1, M_2, M_3\}$, where $M_0=\{4, 0, 0\}$, $M_1=\{1, 3, 0\}$, $M_2=\{0, 2, 2\}$, and $M_3=\{1, 0, 3\}$.

Some of the properties of Generalized Cyclic designs in factorial experiments have been investigated by John (1973). Of primary interest was the problem of obtaining a group of non-isomorphic sets, that is a group such that one set cannot be obtained from another set in the group merely by relabelling the treatments. Some results on the construction of fractional sets also have been obtained. Furthermore, he considered the efficiency of the factorial effects in the Generalized Cyclic designs, and in particular it has been shown how to obtain designs that maximize the efficiency of the main effects.

6. Property C and Property A in Block Designs

In an incomplete block experiment, Kurkjian and Zelen (1963) introduced a structural property of the design related to the block incidence matrix N of the design. This structural property was termed Property A: A block design is said to have Property A, if

$$NN' = \sum_{s=0}^{n} \left\{ \sum_{\delta_1+\delta_2+\cdots+\delta_n=s} h(\delta_1, \delta_2, \cdots, \delta_n) \left(\prod_{i=1}^{n} \otimes D_i^{\delta_i} \right) \right\}, \tag{6.1}$$

where $\delta_i=0$ or 1 for $i=1,2,\cdots,n$, and $h(\delta_1, \delta_2, \cdots, \delta_n)$ are constants, and where $D_i^{\delta_i}$ is I_{m_i} if $\delta_i=0$ and $J_{m_i \times m_i}$ if $\delta_i=1$. Since $D_i^0=R_{m_i}^0$ and $D_i^1=R_{m_i}^0+R_{m_i}^1+\cdots+R_{m_j}^{m_i-1}$, $D_i^{\delta_i}$ is a circulant matrix, so the matrix NN' in (6.1) is an nth order nested block circulant matrix. Shah (1960) considered the following association scheme: The two treatment combinations are the $(p_1, p_2, \cdots, p_n)$th associates, where $p_i=1$ if the ith factor occur at the same level in both treatment combinations and $p_i=0$ otherwise. It has been shown that a Balanced Factorial Experiment is a PBIB design with respect to the above association scheme. Paik and Federer (1973) designated the association scheme as Binary Number Association Scheme (BNAS). The Group Divisible designs, Rectangular designs, Hierarchical Group Divisible designs, and the Direct Product designs are BNAS-PBIB designs, but most Cyclic and Generalized Cyclic designs are not so. They are CFAS-PBIB designs. Since every incomplete block design having Property A is a PBIB design with BNAS, any PBIB design having BNAS is a special case of PBIB design with CFAS.

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〈**Corrections**〉

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Corrections to "Cyclic Factorial Association Scheme Partially Balanced Incomplete Block Designs"

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The following corrections should be made:

On page 35, in the formulas (4.10) and (4.11), the expression $\prod_{k=1}^{n} \cos(2 i_k j_k \pi / m_k)$ should be replaced by

$$\cos\left(\sum_{k=1}^{n} 2 i_k j_k \pi / m_k\right).$$

References

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