論 文
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變形된 結合係数 디지탈 필터의 安定度 解析

Stability Analysis of Modified Coupled-Form Digital Filter Using a Constructive Algorithm

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요 약

Brayton-Tong의 컴퓨터 앨고리즘을 사용하여 差分方程式으로 표시되는 시스템의 安定度를 판별한다. 특히, 디지탈 信號의 量子化또는 잉여현상(quantization 또는 overflow)에 기인한 非線型 디지탈 필터中에서, 變形된 結合係数 디지탈 필터의 安定度 영역을 계수 공간에서 구한다. 이 영역에 존재하는 계수를 가진 디지탈 필터는 리미트 사이클(Limit Cycle)을 갖지 않는다. Jury-Lee의 절대 안정도 판별 방법보다 개선된 결과를 얻었다.

Abstract

Using the constructive algorithm proposed by Brayton and Tong, we analyze the stability of a modified coupled-form digital filter with quantization and overflow nonlinearities, and find the regions in the parameter plane where the filter is globally asymptotically stable. In these regions, the absence of zero-input limit cycles is ensured. This constructive algorithm gives less conservative stability results than the application of Jury-Lee stability criterion does.

1. Introduction

Digital filters are often implemented using a microprocessor with fixed-point arithmetic. Due to the finiteness of the signal wordlength, digital filters become nonlinear[1], and for this reason the output of the filter deviates from what is actually desired.

In two papers[2] and [3], Brayton and Tong established some significant results which make it possible to construct computer-generated Lyapunov functions to analyze the stability of nonlinear systems. In this paper, we find the regions in the parameter plane where a modified coupled-form digital filter is globally asymptotically stable using the constructive algorithm. The authors of [4] suggested this specific digital filter for further study.

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2. Constructive Stability Algorithm

In this section, we show how the constructive algorithm can be applied to the stability analysis of systems described by a set of difference equations

$$x(k+1)=g[x(k)] \qquad (E)$$

where $x \in R^n, g:R^n \rightarrow R^n, k=0,1,2,\cdots$, and R^n denotes the set of real-valued n-tuples.

In [2] and [3], a set $\underline{A}$ of matrices is said to be stable if there exists a bounded neighborhood of the origin $W \subset R^n$ such that $MW \subseteq W$ for every $M \in \underline{A}$. Equivalently, there exists a vector norm $|\cdot|_w$ such that $|Mx|_w \leq |x|_w$ for all $M \in \underline{A}$ and $x \in R^n$. Therefore, $v = |x|_w$ defines a Lyapunov function for $\underline{A}$. Next, a set $\underline{A}$ of matrices is said to be asymptotically stable if there exists a number $\rho > 1$ such that $\rho\underline{A}$ is stable.

In [2] and [3], a constructive algorithm is given to determine whether a set of m real matrices $\underline{A} = \{M_0, \cdots, M_{m-1}\}$ is stable by starting with an initial polyhedral neighborhood of the origin W_0 and by defining a sequence of regions W_{k+1} by

$$W_{k+1} = H[\bigcup_{j=0}^{\infty} M_i^j W_k], \text{ where } i = (k-1) \bmod m, \quad (1)$$

and where H[W] is the convex hull of W. Now $\underline{A}$ is stable if and only if

$$W^* = \bigcup_{k=0}^{\infty} W_k \quad (2)$$

is bounded.

To utilize the constructive algorithm, we rewrite the given system equation (E) as

$$x(k+1) = M[x(k)]x(k) \quad (3)$$

where $M[x(k)]$ is chosen so that $M(x)x = g(x)$. If we let $\underline{M}$ denote the set of all matrices obtained by varying x in M(x) over all allowable values, then we can rewrite (3) equivalently as

$$x(k+1) = M_k x(k), \; M_k \in \underline{M}. \quad (4)$$

In [2] and [3], it is shown that the equilibrium $x=0$ of (E) is globally asymptotically stable if the set of matrices $\underline{M}$ is asymptotically stable.

We will call any nontrivial periodic solution of (E) a limit cycle. Note that if a system (E) is globally asymptotically stable, then no limit cycles will exist for system (E).

For those terms related to stability, refer to [5]. For the concept of an extreme matrices in a linear vector space of real matrices, refer to [6]. Also refer to [2], [3] and [7] for further details of the constructive algorithm.

3. Nonlinearities in Digital Filters

In fixed-point arithmetic, quantization can be performed by substituting the nearest possible number that can be represented by the limited number of bits. This type of nonlinear operation is called a round off quantizer. Another possiblity consists of discarding the least significant bits in the number. If the signals are represented by sign and magnitude then we have a magnitude truncation quantization nonlinearity.

If an overflow occurs, a number of different actions may be taken. If the number that caused the overflow is replaced by a number having the same sign, but with a magnitude corresponding to the overflow level, saturation overflow is obtained. Zeroing overflow substitutes the number zero in case of an overflow. In two's complement arithmetic, the most significant bits that caused the overflow are discarded. In this case two's cnmplement overflow is used, and overflow in intermediate results do not cause errors, as long as the final result does not have overflow. Another way of dealing with overflow is the triangular overflow as proposed by Eckhardt and Winkelnkemper(see[1]). For the details and characteristics of these nonlinearities, refer to [1] and [8].

It is possible to have different wordlengths for the various signals in the filter, resulting in different quantization stepsizes and/or different overflow levels. We will assume throughout this paper that all quantizers in a filter have the same quantization stepsize, q, and are the same type, e.g., round off or truncation. Similarly, we will assume that all overflow nonlinearities in a filter have the same overflow level, p, and are the same type.

The above nonlinearities will be viewed as belonging to a sector $[k_m, k_M]$, where

$$k_m \leq f(w)/w \leq k_M \text{ for all } w \in R. \quad (5)$$

Under the above assumptions, we view the quantization nonlinearities as belonging to the sector $[0, k_q]$ where

$$k_q = \begin{cases} 1 & \text{for truncation} \\ 2 & \text{for roundoff.} \end{cases} \quad (6)$$

Henceforth, k_q will represent the upper slope of the sector that contains the quantization nonlinearity. The overflow nonlinearilies are represented as belonging to the sector $[k_o, 1]$ where

$$k_o = \begin{cases} 0 & \text{saturation or zeroing} \\ -1/3 & \text{triangular} \\ -1 & \text{two's complement.} \end{cases} \tag{7}$$

Henceforth, k_o will represent the lower slope of the sector that contains the overflow nonlinearity.

4. Modified Coupled-Form Digital Filter

Yan and Mitra [9] proposed two variations of the well-known coupled form. These new digital filter structures have lower pole sensitivites and roundoff noise variances than those of the coupled form and have been derived using the netwowork transformation approach of Szczupak and Mitra [10]. There are two structures, first and second. In this paper, second structure will be employed. The linear filter is globally asymptotically stable if and only if its poles lie within the region in which the parameters a and b must satisfy.

$$b^2(1 + a^2) < 1. \tag{8}$$

Quantization is assumed to take place after each multiplication and overflow is placed after each addition. This structure is realistic and is shown in Fig. 1.

4.1 Constructive Stability Results

The state equations for the structure with four quantizers in Fig. 1 are

$$x_1(k+1) = P_1[Q_1(bx_2(k)) + Q_3[aQ_2(bx_1(k))]] \tag{9}$$
$$x_2(k+1) = P_2[-Q_2(bx_1(k)) + Q_4[aQ_1(bx_2(k))]]$$

with Q_i, $i=1, 2, 3, 4$ representing the quantizers and P_i, $i=1,2$ represening the overflow nonlinearities, which satisfy the sector conditions (6) and (7), respectively. The state equations are written as

$$x(k+1) = M[x(k)]x(k). \tag{10}$$

By defining

$$\phi_1(x) = \frac{P_1[Q_1(bx_2)+Q_3(aQ_2(bx_1))]}{Q_1(bx_2)+Q_3(aQ_2(bx_1)}$$

$$\phi_2(x) = \frac{Q_1(bx_2)}{bx_2}$$

$$\phi_3(x) = \frac{Q_3[aQ_2(bx_1)]}{aQ_2(bx_1)}$$

$$\phi_4(x) = \frac{Q_2(bx_1)}{bx_1} \tag{11}$$

$$\phi_5(x) = \frac{P_2[-Q_2(bx_1)+Q_4[aQ_1(bx_2)]]}{-Q_2(bx_1)+Q_4[aQ_1(bx_2)]}$$

$$\phi_6(x) = \frac{Q_4[aQ_1(bx_2)]}{aQ_1(bx_2)}$$

the matrix $M(x(k))$ is given by

$$M(x(k)) = \begin{bmatrix} ab\phi_7(x) & b\phi_8(x) \\ -b\phi_9(x) & ab\phi_{10}(x) \end{bmatrix} \tag{12}$$

where $\phi_7(x) = \phi_1(x)\ \phi_3(x)\ \phi_4(x)$,
$\phi_8(x) = \phi_1(x)\ \phi_2(x)$
$\phi_9(x) = \phi_4(x)\ \phi_5(x)$ and
$\phi_{10}(x) = \phi_2(x)\ \phi_5(x)\ \phi_6(x)$.

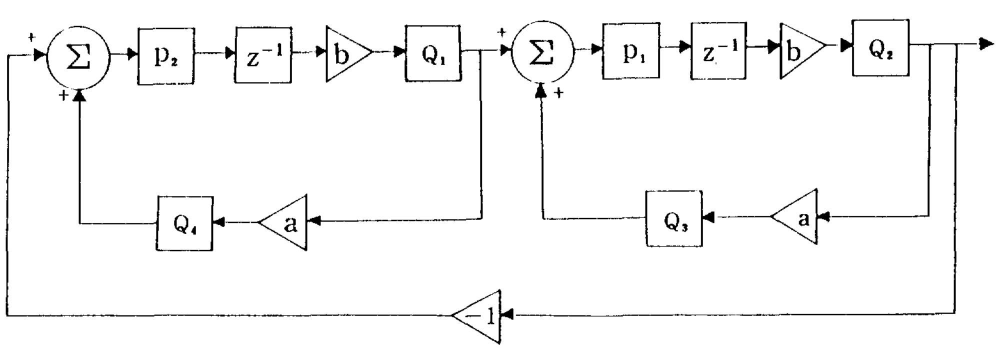

Fig. 1. Modified coupled-form with four quantizers-second structure.

The functions $\phi_i(x)$, i=7, 8, 9, 10 are bounded by constants ;

$$\begin{aligned} c_1 \leqq \phi_i(x) \leqq c_2, \quad & i=8, 9 \\ d_1 \leqq \phi_i(x) \leqq d_2, \quad & i=7, 10 \end{aligned} \tag{13}$$

where

$$c_1 = k_o k_q^2$$
$$c_2 = k_q^2$$
$$d_1 = k_o k_q$$
$$d_2 = k_q \cdot$$

Thus, the extreme matrices of the set $\underline{M}$ are

$$E(\underline{M}) = \left\{ \left[\begin{matrix} abc_i & bd_j \\ -bc_k & abd_m \end{matrix} \right] \right\}, i, j, k, m=1, 2 \Big\} \cdot \tag{14}$$

In this case, the constructive algorithm uses sixteen extreme matrices for every point in the a-b parameter plane. If the overflow nonlinearities are absent, then the set of extreme matrices in this case is the same as for the filter with four saturation or zeroing overflow nonlinearities.

Using these extreme matrices, we can get the stability results by applying the constructive algorithm. Some of those results are shown in Fig. 2-4. Only half of these regions are shown, since they are symmetric about the b-axis. We used the value of $\rho=1.001$ to show that A is asymptotically stable for a stable matrix A.

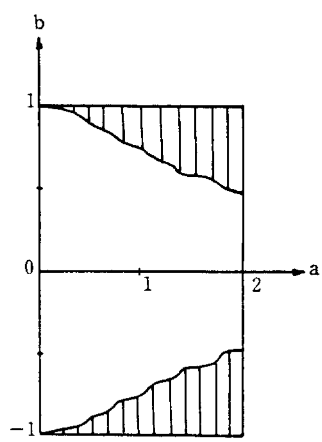

Fig. 2. Stability region for magnitude truncation quantizer and saturatino or zeroing overflow.

4.2 Jury and Lee Stability Results

For the digital filters with quantizers and no overflow nonlinearities, an absolute stability criterion by Jury and Lee[11] can be used to determine sufficient conditions for the global asymptotic stability of the equilibrium of the system.

A system with several nonlinearities is represented by the system shown in Fig. 5. The m nonlinear elements are represented by the vector-valued function f(w) where $f_i(w_i)$ is the output

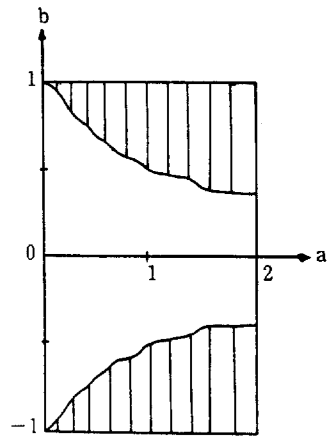

Fig. 3. Stability region for magnitude truncation quantizer and two's complement overflow.

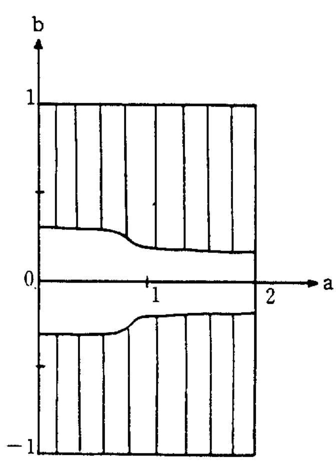

Fig. 4. Stability region for roundoff quantizer and triangular overflow.

of the i-th nonlinear element. The input of this element is the i-th component of the vector $w^T = [w_1, \cdots, w_m]$.

The inputs and outputs of the nonlinear elements are interconnected by linear filters with transfer functions, $g_{ij}(z)$, which are assumed to be controllable and observable[12], that are the elements of the m x m transfer matrix G(z). The linear filter $g_{ij}(z)$ connects the output of the j-th nonlinear element and the input of the i-th nonlinear element. We assume that each element $g_{ij}(z)$ has all of its poles within the unit circle except possibly one pole at $z=1$. We assume that the nonlinearities $f_i(w_i)$ satisfy the following conditions:

$$\begin{aligned} &\text{i)}\ f_i(0)=0 \\ &\text{ii)}\ 0 < f_i(w_i)/w_i < k_{ii},\ \text{for } w_i \neq 0 \qquad (15) \\ &\text{iii)}\ w(k)\to 0 \text{ implies } y(k)\to 0 \\ &\text{iv)}\ -\infty < df_i(w_i)/dw_i < \infty,\ i=1,2,\cdots,m \end{aligned}$$

where k_{ii} is the i-th diagonal element of the m x m matrix K.

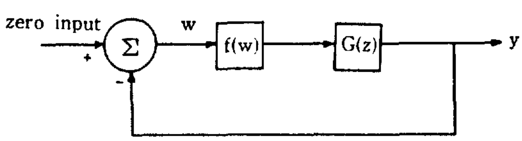

Fig. 5. A general system with many nonlinearities.

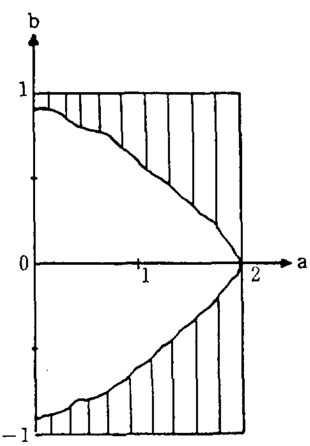

Fig. 6. Stability region for maginitude truncation quantizer by theorem 1.

Theorem 1 [11] : The system of Fig.5 satisfying the above conditions for G(z) with nonlinearities described by (15) is globally asymptotically stable if

$$H(z)=2K^{-1}+G(z)+G^*(z) \qquad (16)$$

is positive definite for all $z: |z|=1$, where $G^*(z)$ denotes the complex conjugate transpose of G(z).

If we apply the Jury and Lee criterion to this filter with four quantizers and no overflow, the matrix G(z) may be written as

$$G(z)=\begin{bmatrix} 0 & bz^{-1} & 0 & -bz^{-1} \\ -bz^{-1} & 0 & -bz^{-1} & 0 \\ 0 & -a & 0 & 0 \\ -a & 0 & 0 & 0 \end{bmatrix} \qquad (17)$$

The matrix H(z), given by

$$H(z)=\begin{bmatrix} 2/k_{11} & -b(z-z^{-1}) & 0 & -(a+bz^{-1}) \\ b(z-z^{-1}) & 2/k_{22} & -(a+bz^{-1}) & 0 \\ 0 & -(a+bz) & 2/k_{33} & 0 \\ -(a+bz) & 0 & 0 & 2/k_{44} \end{bmatrix} \qquad (18)$$

must be positive definite for all $z:|z|=1$. For magnitude truncation quantizers, $k_{ii}=1$ and for roundoff quantizers, $k_{ii}=2$, $i=1, 2, 3, 4$. The region of the global asymptotic stability in the parameter plane for the case of the magnitude truncation quantizers is shown in Fig. 6. Only half of the region is shown, since it is symmetric about the b-axis.

As can be seen from Figures 2 and 6 for the digital filter with quantizer and no overflow nonlinearities, the constructive algorithm gives less conservative results than the application of the Jury and Lee stability criterion. All of the results obtained seem to be new. Those regions are depicted as the unhatched regions in the figures.

5. Conclusion

Using the constructive stability algorithm proposed by Brayton-Tong, we analyzed the stability of the equilibrium $x=0$ of a modified coupled-form digital filter described by a set of the difference equations. All the results seem to be new. Also we used the Jury and Lee absolute stability criterion for comparison with the constructive results. Both results yield sufficient conditions for global asymptotic stability in terms of the parameters of a given filter under zero-input conditions. These results constitute also sufficient conditions for the absence of zero-input limit cycles.

While existing methods[1] [8] of stability analysis are generally different for each particular structure, the constructive algorithm allows us to use one method to study the stability of nonlinear digital filtes , and moreover it may be applied to higher order filters by considering the higher order filter as an interconnection of lower order structures. There are many other digital filter structures such as universal CGIC filters that could also be studied.

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