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ON EXISTENCE OF FIXED POINTS IN 2-METRIC SPAECS

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In this paper, several fixed point theorems for a pair of mappings of a (S, T)-orbitally complete 2-metric space into itself are proved. In [2] and [6], L.B. Ciric and R.E. Smithson introduced the concepts of orbital completeness and orbital continuity of mappings on metric spaces and K. Iseki extended the concepts of orbital completeness and orbital continuity of mappings on metric spaces and K. Iseki extended the concepts of orbital completeness and orbital continuity to 2-metric spaces ([3]). Especially L.B. Ciric proved the following theorem ([2]) : Let (X, d) be a T-orbitally complete metric space and a mapping T of X into itself be orbitally continuous. If T satisfies the following condition:

$$\min\{d(Tx, Ty), d(x, Tx), d(y, Ty)\} - \min\{d(x, Ty), d(y, Tx)\} \leqslant \alpha d(x, y)$$

for every x, y in X and for some α $(0<\alpha<1)$, then for each x_0 in X, the sequence $\{T^n x_0\}$ converges to a fixed point of T.

Motivated by this result of L.B. Ciric, J. Achari ([1]), K. Iseki ([4]) and S.N. Mishra ([5]) extended this result to multivalued mappings on metric spaces and 2-metric spaces, respectively.

Before going our main theorems, we need the following definitons:

DEFINITION 1. In a 2-metric space (X, d), if $d(x_n, x, z)$ converges to 0 for all z in X as $n \to \infty$, we say that a sequence $\{x_n\}$ in X converges to x and x is called the limit of this sequence.

DEFINITION 2. In a 2-metric space (X, d), if $d(x_m, x_n, z)$ converges to 0 for all z in X as $m, n \to \infty$, we say that a sequence $\{x_n\}$ in X is called a Cauchy sequence in X. If every Cauchy sequence is convergent, X is called complete.

DEFINITION 3. Let S and T be two mappings from a 2-metric space (X, d) into itself. For any x_0 in X, a sequence $0(x_0, S, T) = \{x_0 = x, \ x_1 = Sx_0, \ x_2 = Tx_1, \cdots, \ x_{2n} = Tx_{2n-1}, \ x_{2n+1} = Sx_{2n}, \cdots\cdots\}$ is called an orbit of S and T at the point x_0 in X.

DEFINITION 4. A 2-metric space (X, d) is (S, T)-orbitally complete if every Cauchy sequence contained in the orbit of S and T at some point converges in X.

If $S = T$ in Definition 3, then $0(x_0, S, T)$ is called an orbit of T at the point x_0 in X, and a 2-metric space (X, d) in which every Cauchy sepuence contained in the orbit $0(x_0, T, T) = 0(x_0, T)$ converges is called T-orbitally complete.

REMARK 1. Every complete 2-metric space is (S, T)-orbitally complete but the converse is not true. For example, Let $X = (0, 1] \times (0, 1]$ and define a 2-metric d on X by

$$d(x,y,z)=\frac{1}{2}\text{abs}\begin{vmatrix} x_1 & x_2 & 1 \\ y_1 & y_2 & 1 \\ z_1 & z_2 & 1 \end{vmatrix}$$

for every $x=(x_1,x_2)$, $y=(y_1,y_2)$ and $z=(z_1, z_2)$ in X. Let $T(x, y)=(\frac{x+1}{2}, \frac{y+1}{2})$ and $S(x, y)=(x, y)$ for every x, y in X. Then a 2-metric space (X,d) is (S,T)-orbitally complete but not complete.

Definition 5. A mapping T of 2-metric space (X,d) into itself is said to be sequentially continuous if for every sequence $\{x_n\}$ such that $d(x_n, x, z)$ converges to 0 for all z in X as $n\to\infty$, $d(Tx_n, Tx, z)$ converges to 0 as $n\to\infty$. A mapping T is called orbitally continuous if for all z in X, $d(T^nx, y, z)$ converges to as $n\to\infty$ implies that $d(T^{n+1}x, Ty, z)$ converges to 0 as $n\to\infty$. Every sequentially continuous mapping is orbitally continuous.

Now we are ready to give our main theorems:

Theorem 6. Let (X,d) be a (S,T)-orbitally complete 2-metric space and mappings S and T of X into itself be sequentially continuous. If mappings S and T satisfy the following condition (A):

$$\text{(A)}\ \min\{d(Sx, Ty, z), d(x, Sx, z), d(y, Ty, z), d(Sx, TSx, z), d(y, TSx, z)\}$$
$$+k\min\{d(x, Ty, z), d(y, Sx, z), d(x, STy, z), d(Ty, TSx, z)\}\leqslant \alpha d(x,y,z)$$

for every x, y, z in X, where $\alpha\in(0,1)$ and k is a real number, then the orbit of S and T at x_0, $0(x_0, S, T)$, converges to a point u in X and u is a common fixed point of S and T. If $k>\alpha$, then S and T have a unique common fixed point in X.

PROOF. For given x_0 in X, consider the orbit of S and T at the point x_0, $0(x_0, S, T)=\{x_0=x, x_1=Sx_0, x_2=Tx_1, \cdots, x_{2n}=Tx_{2n-1}, x_{2n+1}=Sx_{2n}, \cdots\cdots\}$. If, for some m, $x_m=x_{m+1}$, then S and T have a common fixed point x_m in X. Thus we suppose that $x_m \neq x_{m+1}$ for all m=1, 2, 3, From the condition (A), for $x=x_{2n}$ and $y=x_{2n+1}$, we have

$$\begin{aligned}&\min\{d(Sx_{2n}, Tx_{2n+1}, z), d(x_{2n}, Sx_{2n}, z),\\ &\quad d(x_{2n+1}, Tx_{2n+1}, z), d(Sx_{2n}, TSx_{2n}, z),\\ &\quad d(x_{2n+1}, TSx_{2n}, z)\}\\ &+k\min\{d(x_{2n}, Tx_{2n+1}, z), d(x_{2n+1}, Sx_{2n}, z),\\ &\quad d(x_{2n}, STx_{2n+1}, z), d(Tx_{2n+1}, TSx_{2n}, z)\}\\ &\leqslant \alpha d(x_{2n}, x_{2n+1}, z)\end{aligned}$$

or

$$\begin{aligned}&\min\{d(x_{2n+1}, x_{2n+2}, z), d(x_{2n}, x_{2n+1}, z)\}\\ &\leqslant \alpha d(x_{2n}, x_{2n+1}, z)\end{aligned}$$

for every non-negative integer n. Since (X,d) is a 2-metric space, $d(x_{2n}, x_{2n+1}, z) \neq 0$ for some z in X. Hence if $d(x_{2n}, x_{2n+1}, z) < d(x_{2n+1}, x_{2n+2}, z)$, then we have $d(x_{2n}, x_{2n+1}, z) \leqslant \alpha d(x_{2n}, x_{2n+1}, z)$ for $\alpha \in (0,1)$, which is impossible and so we have $d(x_{2n+1}, x_{2n+2}, z) \leqslant \alpha d(x_2, x_{n+1}, z)$. Similarly, we have $d(x_{2n}, x_{2n+1}, z) \leqslant \alpha d(x_{2n-1}, x_{2n}, z)$. Therefore $d(x_m, x_{m+1}, z) \leqslant \alpha d(x_{m-1}, x_m, z)$ for every non-negative integer m and hence since $d(x_0, x_1, x_m) \leqslant d(x_0, x_1, x_{m-1}) + d(x_0, x_{m-1}, x_m) + d(x_{m-1}, x_1, x_m) = d(x_0, x_1, x_{m-1}) + d(x_m, x_{m-1}, x_2) + d(x_m, x_{m-1}, x_1)$, we have

$$\begin{aligned}d(x_0, x_1, x_m) &\leqslant d(x_0, x_1, x_{m-1}) + \alpha^{m-1}(d(x_1, x_0, x_0)\\ &+d(x_1, x_0, x_1)) = d(x_0, x_1, x_{m-1})\\ &\leqslant d(x_0, x_1, x_{m-2})\end{aligned}$$

$$\begin{aligned}&\leqslant\cdots\cdots\\&\leqslant d(x_0,\ x_1,\ x_2)\\&\leqslant \alpha d(x_0,\ x_0,\ x_1)\\&=0.\end{aligned}$$

Therefore, from $d(x_m,\ x_n,\ z)\leqslant \sum_{k=0}^{m-n-2} d(x_m,\ x_{n+k},\ x_{n+k+1})+\sum_{k=0}^{m-n-1} d(x_{n+k},\ x_{n+k+1},\ z) < \frac{\alpha^n}{1-\alpha}(d(x_0,\ x_1,\ z)+d(x_0,\ x_1,\ z))$, it follows that $\{x_n\}$ is a Cauchy sequence. Since $(X,\ d)$ is a (S,T)-orbitally complete, $\{x_n\}$ converges to some point u in X. Using the sequential continuity of S, we have

$$\begin{aligned}0&\leqslant d(u,\ Su,\ z)\\&\leqslant d(u,\ Su,\ x_{2n+1})+d(u,\ x_{2n+1},z)+d(x_{2n+1},\ Su,\ z)\\&=d(u,\ Su,\ Sx_{2n})+d(u,\ x_{2n+1},\ z)+d(Sx_{2n},\ Su,\ z)\\&\rightarrow 0 \text{ as } n\rightarrow\infty.\end{aligned}$$

Therefore $d(u,\ Su,\ z)=0$ for all z in X. Thus u is a fixed point of S. Similarly, u is also a fixed point of T, that is, u is a common fixed point of S and T. Next, let $k>\alpha$ and to prove the uniqueness of a common fixed point of S and T, let u and v be common fixed points of S and T. Since (X,d) is a 2-metrics pace, there exists a point w in X such that $d(v,\ u,\ w)\neq 0$, For this w in X,

$$\begin{aligned}&\min\{d(Sv,\ Tu,\ w),\ d(v,\ Sv,\ w),\ d(u,\ Tu,\ w),\\&\qquad d(Sv,\ TSv,\ w),\ d(u,\ TSv,\ w)\}\\&+k\min\{d(v,\ Tu,w),\ d(u,\ Sv,\ w),\ d(v,\ STu,w),\\&\qquad d(Tu,\ TSv,\ w)\}\\&\quad\leqslant \alpha d(v,\ u,\ w)\end{aligned}$$

gives

$$d(v, u, w) \leqslant \frac{\alpha}{k} d(v, u, w),$$

which is impossible. Therefore $u=v$. This proves that S and T have a unique common fixed point u in X.

REMARK 2. In Theorem 6, if we take $S=T$, we have a sequence $\{x_n\}$, where $x_n=T^n x_0$ such that $d(x_n, u, z)$ converges to 0 as $n\to\infty$. Since T is sequentially continuous, it is orbitally continuous. Hence we have $d(Tx_n, Tu, z)$ converges to 0 as $n\to\infty$. Thus, in Theorem 6, if $T=S$ is orbitally continuous, $\{T^n x_0\}$ converges to a fixed point u of T. If $k=-1$ and (X,d) is a bounded complete 2-metric space, this result is similar to the result of S.N. Mishra ([5]).

COROLLARY 7. Let (X,d) be a (S, T)-orbitally complete 2-metric space. If mappings S and T of X into itself satisfy the following condition (B):

$$\text{(B)} \quad \min\{d(Sx, Ty, z), d(x, Sx, z), d(y, Ty, z)\} + k\min\{d(x, Ty, z), d(y, Sx, z)\} \leqslant \alpha d(x, y, z)$$

for every x, y, z in X, where $\alpha \in (0,1)$ and k is a real number, then the orbit of S and T at x_0, $0(x_0, S, T)$, converges to a point u in X and u is a common fixed point of S and T. If $k>\alpha$, then S and T have a unique common fixed point in X.

We note that in Corollary 7 if we put $S=T$ and $k=-1$ and if (X,d) is a bounded complete 2-metric space and T is orbitally continuous, then we have the result of K. Iseki ([3]), which extends the result of L.B. Ciric([2]) to a 2-metric space.

For any positive integer powers of S and T, we have the following theorem from Theorem 6:

THEOREM 8. Let (X,d) be a (S,T)-orbitally complete 2-metric space and mappings S and T of X into itself be sequentially continuous. If mappings S and T satisfy the following condition (C):

$$\begin{aligned}(C)\quad & \min\{d(S^s x, T^t y, z), d(x, S^s x, z), d(y, T^t y, z), \\ & \qquad d(y, T^t S^s x, z), d(y, T^t S^s x, z)\} \\ & +k\min\{d(x, T^t y, z), d(y, S^s x, z), d(x, S^s T^t y, z), \\ & \qquad d(T^t y, T^t S^s x, z)\} \\ & \leqslant \alpha d(x, y, z),\end{aligned}$$

for every x, y, z in X and for some positive integers s and t, where $\alpha \in (0,1)$ and k is a real number such that $k > \alpha$, then S and T have a unique common fixed point in X.

PROOF. If we take $P = S^s$ and $Q = T^t$, by Theorem 6, P and Q have a unique common fixed point u in X, that is, $Pu = Qu = u$. From this,

$$Su = PSu = QSu \text{ and } Tu = PTu = QTu,$$

that is, Su and Tu are common fixed points of P and Q. If we put $x = Su$ and $y = Tu$ in the condition (C), we obtain $d(Su, Tu, z) \leqslant \alpha d(Su, Tu, z)$ for $\alpha \in (0,1)$, which means $Su = Tu$. Therefore the uniqueness of u implies that $u = Su = Tu$, that is, u is a unique common fixed point of S and T.

THEOREM 9. Let (X,d) be a (S,T)-orbitally complete 2-metric space and $\{S_n\}$ and $\{T_n\}$ be sequences of sequentially continuous mappings of X into itself and let $\{s_q\}$

and $\{t_p\}$ be sequences of positive integers such that for any positive integers pair p, q $(p\neq q)$,

$$\min\{d(S_p^{s_p}(x),\ T_q^{t_q}(y),\ z),\ d(x, S_p^{s_p}(x), z),$$
$$d(y,\ T_q^{t_q}(y),\ z),\ d(S_p^{s_p}(x),\ T_q^{t_q}S_p^{s_p}(x),\ z),$$
$$d(y,\ T_q^{t_q}S_p^{s_p}(x),\ z)\}$$
$$+k\min\{d(x,\ T_q^{t_q}(y),\ z),\ d(y, S_p^{s_p}(x),\ z),$$
$$d(x,\ S_p^{s_p}T_q^{t_q}(y), z),\ d(T_p^{t_p}(y),\ T_q^{t_q}S_p^{s_p}(x), z)\}$$
$$\leqslant \alpha d(x,y,z),$$

for every x, y, z in X, where $\alpha\in(0,1)$ and k is a real number such that $k>\alpha$. Then the sequences $\{S_n\}$ and $\{T_n\}$ have a unique common fixed point u in X.

PROOF. If we take any pair of positive ~~integers pair~~ p and q $(p\neq q)$, then, by Theorem 8, S_p and T_q have a unique common fixed point in X. Since p and q are arbitrary, this theorem follows.

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