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SOME REMARKS ON ISOLATED SINGULARITIES

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1. Introduction

Let $D(a;r)$ be the open disc with center at a and radius r in the complex plane, and $D'(a;r)$ be the punctured disc with center at a and radius r. We denote by $H(G)$ the class of all holomorphic functions in a plane open set G. The letter G will from now on denote a plane open set.

DIFINITION. *If* $a \in G$ and $f \in H(G-\{a\})$, then f is said to have an isolated singularity at the point a. If f can be so defined at a that the extended function is holomorphic in G the singularity is said to be removable.

If $a \in G$ and $f \in H(G-\{a\})$, then one of the following three cases must occur[4, p. 227]:

(a) f has a removable singularity at a.

(b) There are complex numders $c_1, \cdots, c_m$, where m is a positive integer and $c_m \neq 0$, such that

$$f(z)-\sum_{k=1}^{m}\frac{c_k}{(z-a)^k}$$

has a removable singularity at a.

(c) *If* $r>0$ and $D(a;r) \subset G$, then $f(D'(a;r))$ is dense in the plane.

In case (b), f is said to have a pole of order m at a. The function $\sum_{k=1}^{m} c_k(z-a)^{-k}$, a polynomial in $(z-a)^{-1}$, is called the principal part of f at a. In case (c), f is said to have an essential singularity at a.

In this note, we investigate some properties of isolated singularities. In section 2 we find simple conditions on f that are equivalent to the statement that f has a removable singularity at a (and similarly for poles and essential singularities). In section 3 we consider the extended complex plane C_∞.

2. Isolated singularities.

We begin with the following theorem.

THEOREM 1. *If* $a \in G$ and $f \in H(G-\{a\})$, then the following statements are equivalent:

(a) f has a removable singularity at a.

(b) $f(z)$ approaches a finite limit as $z \to a$.

(c) $\lim_{z \to a} (z-a)f(z)=0$.

(d) The Laurent expansion of f about a has no negative powers.

PROOF. (a) implies (b): Let g be the holomorphic extention of f. Since g is continuous at a, it follows that

$$\lim_{z \to a} f(z)=\lim_{z \to a} g(z)=g(a);$$

hence $f(z)$ approaches a finite limit as $z \to a$.

(b) implies (c): Obvious.

(c) implies (d): The function g defined in G by

$$g(z)=\begin{cases}(z-a)\ f(z) & if\ z \neq a, \\ 0 & if\ z=a,\end{cases}$$

is continuous in G and holomorphic in $G-\{a\}$. Then it follows from a theorem [2, p. 31] that $g \in H(G)$. Thus

$$g(z) = \sum_{n=1}^{\infty} c_n (z-a)^n \ (z \in D\,(a;r) \subset G).$$

Consequently we have

$$f(z) = \sum_{n=0}^{\infty} c_{\ +1} (z-a)^n \ \ (z \in D'(a;r) \subset G).$$

(d) implies (a): Let $f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$ be its Laurent expansion in $D'\,(a;r) \subset G$. Then the function g defined in G by

$$g(z) = \begin{cases} f(z) & if\ z \neq a, \\ c_0 & if\ z = a, \end{cases}$$

is holomorphic in G and agrees with f in $G - \{a\}$.

We consider now the characterization of poles.

THEOREM 2. *If* $a \in G$ and $f \in H(G - \{a\})$, then the following statements are equivalent:

(a) f has a pole at a.

(b) $\lim_{z \to a} f(z) = \infty$.

(c) There is a positive integer m and a $g \in H(G)$ with $g(a) \neq 0$ such that $f(z) = (z-a)^{-m}\, g(z)$.

(d) There is a positive integer m such that $(z-a)^m f(z)$ approaches a finite nonzero limit as $z \to a$.

(e) The Laurent expansion of f about a has a positive but finite number of negative powers.

PROOF. (a) implies (b): Let $\sum_{k=1}^{m} c_k\,(z-a)^{-k}$ be the principal part of f at a and let g be the holomorphic extension of $f(z-a)^{-k}$. Since $c_m \neq 0$ and g is continuous at a, it follows that

$$\lim_{z\to a} f(z) = \lim_{z\to a} \sum_{k=1}^{m} c_k (z-a)^{-k} + g(z) = \infty.$$

(b) implies (c): Let M be a positive real number. Since $\lim_{z\to a} f(z)=\infty$, there exists $r>0$ such that $D(a;r) \subset G$ and $|f(z)| \geq M$ whenever $z \in D'(a;r)$. Then $1/f \in H(D'(a;r))$ and $\lim_{z\to a} [f(z)]^{-1} = 0$. Hence, $h(z)=[f(z)]^{-1}$ for $z \neq a$ and $h(a) = 0$, is holomorphic in $D(a;r)$. However, since $h(a) = 0$ it follows that $h(z) = (z-a)\ h_1(z)$ for some $h_1 \in H(D(a;r))$ with h_1 (a)$\neq 0$ and some integer $m \geq 1$. Define $g(a)=1/h_1(a)$, and $g(z)=(z-a)^m f(z)$ in $G-\{a\}$. Then $g \in H(G)$, $f(z)=(z-a)^{-m} g(z)$, and $g(a) \neq 0$.

(c) implies (d) : Obvious.

(d) implies (e) : *If* $(z-a)^m f(z)$ approaches a finite nonzero limit, then $(z-a)^m f(z)$ has a removable singularity at a by THEOREM 1.

Hence there exists $r>0$ such that $D(a;r) \subset G$ and

$$(z-a)^m f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n \quad (c_0 \neq 0, z \in D'(a;r)),$$

so we find upon dividing by $(z-a)^m$ that the Laurent expansion of f about a has a positive but finite number of negative powers.

(e) implies (a) : Let $f(z) = \sum_{n=-m}^{\infty} c_n (z-a)^n$ ($c_{-m} \neq 0$) be its

Laurent expansion in $D'(a;r) \subset G$. Define $g(a) = c_0$, and $g(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$ in $D'(a;r)$. Then $g \in H(D(a;r))$, and hence

has a removable singularity at a.

For a more detailed discussion of isolated singularities, we consider the conditions

$$(A) \ \lim_{z \to a} |z-a|^s f(z) = 0,$$

$$(B) \ \lim_{z \to a} |z-a|^s f(z) = \infty,$$

where s is some real number.

LEMMA 3. Let f have an isolated singularity at a and suppose $f \not\equiv 0$. If either (A) or (B) holds for some real numder s, then there is an integer m such that (A) holds if $s > m$ and (B) holds if $s < m$; furthermore, f has a removable singularity at a if $m \leq 0$ and has a pole at a if $m > 0$.

PROOF. If (A) holds for a certain s, then it holds for all larger s, and hence for some integer p. Then $(z-a)^p f(z)$ has a removable singularity at a. Suppose $f \in H(D'(a;r))$, and let g be the holomorphic extension of $(z-a)^p f(z)$. Since $g(a)=0$ and $g \not\equiv 0$, there exists a unique positive integer k such that

$$g(z) = (z-a)^k g_1(z) \quad (z \in D(a;r))$$

where $g_1 \in H(D(a;r))$ and $g_1(a) \neq 0$. Hence we have

$$(1) \ \lim_{z \to a} |z-a|^s f(z) = \lim_{z \to a} |(z-a)^{s+k-p} g_1(z)|$$

$$=\begin{cases} 0 & \text{if } s>p-k \\ \infty & \text{if } s<p-k. \end{cases}$$

Thus (A) holds for all $s>m=p-k$, while (B) holds for all $s<m$.

Assume now that (B) holds for some s; then it holds for all smaller s, and hence for some integer n. By Theorem 2, there is a positive integer p and a $g_2 \in H(D(a; r))$ with $g_2(a) \neq 0$ such that

$$(z-a)^n f(z)=(z-a)^{-p} g_2(z).$$

Put $m=n+p$. Then we have

$$(2)\ \lim_{z\to a} |z-a|^s |f(z)|=\lim_{z\to a} |(z-a)^{s-m} g_2(z)|$$

$$=\begin{cases} 0 & \text{if } s>m \\ \infty & \text{if } s<m. \end{cases}$$

Finally, suppose $m \leq 0$. Then, by (1) and (2), we have

$$\lim_{z\to a}(z-a)f(z)=\lim_{z\to a}(z-a)^{1-m} g_i(z)=0 \quad (i=1,2),$$

and hence f has a removable singularity at a. *If* $m>0$, then

$$\lim_{z\to a} f(z)=\lim_{z\to a}(z-a)^{-m} g_i(z)=\infty \quad (i=1,2).$$

Hence f has a pole at a.

THEOREM 4. *If* $a \in G$ and $f \in H(G-\{a\})$, then the following statements are equivalent:

(a) f has an essential singularity at a.

(b) $f(z)$ does not approach a finite or infinite limit as $z \to a$.

(c) The Laurent expansion of f about a has an infinite number of negative powers.

(d) Neither $\lim_{z\to a}|z-a|^s|f(z)|= 0$ nor $\lim_{z\to a}|z-a|^s|f(z)|=\infty$

holds for any real number s.

(e) To each complex number w there corresponds a sequence $\{z_n\}$ such that $z_n\to a$ and $f(z_n)\to w$ as $n\to\infty$.

Proof. The equivalence of (a), (b)and (c) follows from Theorem 1 and Theorem 2. By Lemma 3, (a) implies (d). And (d) implies (a), by Theorem 1 and Theorem 2. Thus it suffices to show that (a) is equivalent to (e).

Suppose (a) holds. Choose $\delta>0$ such that $D(a;1/\delta)\subset G$. Then $D(a;1/\delta+n)\subset G$ for $n=1,2,\cdots$. Choose $w_n\in f(D'(a;1/\delta+n))\cap D(w;1/n)$, and choose $z_n\in D'(a;1/\delta+n)$ such that $f(z_n)=w_n$. Then $z_n\to a$ and $f(z_n)\to w$; hence (e) holds.

Conversely, assume that (e) holds. Suppose $D(a;r)\subset G$. Let U be a nonempty open set, and choose a point $w\in U$ with $w\neq f(a)$. Choose $\delta>0$ such that $D(w;\delta)\subset U$. Let $\{z_n\}$ be a sequence such that $z_n\to a$ and $f(z\ )\to w$. (Since $w\neq f(a)$, $z_n\neq a$ for infinitely many n). Then there exists an integer N such that $0<|z_N-a|<r$ and $|f(z_N)-w|<\delta$. Thus $f(z_N)\in f(D'(a;r))\cap U$. Since U is anarbitrary open set, it follows that $f(D'(a;r))$ is dense.

3. The extended complex plane.

For many purpose it is useful to extend the system $\mathbf{C}$ of complex numbers by introduction of a symbol ∞ to represent infinity. For any $r>0$, let $D'(\infty;r)$ be the set of all complex numbers z such that $|z|>r$, put $D(\infty;r)=D'(\infty;r)\cup\{\infty\}$. The set $\mathbf{C}_\infty=\mathbf{C}\cup\{\infty\}$ is topologized in the following manner:

DEFINITION. A subset of $\mathbf{C}_\infty$ is open if and only if it is the union of discs $D(a;r)$, where the a's are arbitrary points of $\mathbf{C}_\infty$ and the r's are arbitrary positive numbers.

THEOREM 5. Let τ be the topology as in the above definition. Then $U \in \tau$ if and only if U is an open subset of $\mathbf{C}$ or $\mathbf{C}_\infty$-U is a closed compact subset of $\mathbf{C}$. That is, the set $\mathbf{C}_\infty$ with the topology τ is the one point compactification of $\mathbf{C}$.

PROOF. Suppose $U \in \tau$. If $\infty \notin U$, it is clear that U is an open subset of $\mathbf{C}$. Suppose $\infty \in U$, and let $U = \bigcup D\ (a;r)$. *If* $a \in \mathbf{C}$, then $D(a;r)^c$ is a closed subset of $\mathbf{C}$. And $D(a;r)^c$ is a closed bounded subset of $\mathbf{C}$ if $a=\infty$. Consequently,

$$\mathbf{C}_\infty - U = \bigcap_{a \neq \infty} D(a;r)^c \cap \bigcap_{a=\infty} U(a;r)^c$$

is a closed bounded subset of $\mathbf{C}$; hence $\mathbf{C}_\infty - U$ is a closed compact subset of $\mathbf{C}$.

Conversely, suppose that U is an open subset of $\mathbf{C}$ or $\mathbf{C}_\infty - U$ is a closed compact subset of $\mathbf{C}$. If U is an open subset of $\mathbf{C}$, it is clear that $U \in \tau$. If $\mathbf{C}_\infty - U$ is a closed compact subset of $\mathbf{C}$, then $\mathbf{C}_\infty - U$ is a bounded subset of $\mathbf{C}$. Thus there exists $r>0$ such that $|z| \leq r$ for every $z \in \mathbf{C}_\infty - U$, and so $D(\infty;r) \subset U$. On the other hand, $\mathbf{C} - (\mathbf{C}_\infty - U) = \mathbf{C} \cap U$ is an open subset of $\mathbf{C}$. Hence $\mathbf{C} \cap U = \bigcup D\ (a;r_a)$, and so

$$\mathrm{U} = (\mathbf{C} \cap U) \cup \{\infty\} = \bigcup D(a;r_a) \bigcup D(\infty;r).$$

Consequently $U \in \tau$.

We note that the extended complex plane $\mathbf{C}_\infty$ is homeomorphic to a sphere. In fact, a homeomorphism φ of $\mathbf{C}_\infty$

onto the unit sphere (where equation in three-dimensional space is $x_1^2 + x_2^2 + x_3^2 = 1$) can be explicitely exhibited : put $\varphi(\infty)=(0,0,1)$. and put

$$\varphi(z) = (2x/|z|^2+1,\ 2y/|z|^2+1,\ |z|^2-1/|z|^2 +1)$$

for all complex numbers $z=x+iy$ [1, p. 18; 3, p. 9]. φ is called a stereographic projection.

The behavior of a complex function f at ∞ may be studied by considering $\tilde{f}(z) = f(1/z)$ at 0. It is clear that $f \in H(D'(\infty;r))$ if and only $\tilde{f} \in H(D'(0;1/r))$. The formal definitions are as follows;

DEFINITION. If f is holomorphic in a punctured disc $D'(\infty;r)$, we say that f has an isolated singularity at ∞. We say that f has a removable singularity, a pole, or an essential singularity at ∞ if $\tilde{f}$ has, respectively, a removable singularity, a pole, or, an essential singularity at 0.

THEOREM 6. Let f be an entire function. Then

(a) f has a removable singularity at ∞ if and only if it is constant.

(b) f has a pole at ∞ of order m if and only if it is a polynomial of degree m.

(c) f has an essential singularity at ∞ if and only if it is not a polynomial.

PROOF. (a) It is clear that every constant function has a removable singularity at ∞. Conversely, suppose that f has a removable singularity at ∞. Since $\tilde{f}$ has a removable singularity at 0, $\tilde{f}(z)$ approaches a finite limit as

$z \to 0$. We define $f(\infty)$ to be this limit, and we thus see that f is entire on $\mathbf{C}_\infty$. Since $\mathbf{C}_\infty$ is compact, f is bounded. Hence, by Liouville's theorem, f is constant.

(b) Suppose f has a pole of order m. Then $\tilde{f}(z)-\sum_{k=1}^{m} c_k z^{-k} (c_m \neq 0)$ has a removable singularity at 0; hence $g(z) = f(z)-\sum_{k=1}^{m} c_k z^k$ has removable singularity at ∞. Since g is entire, it follows from (a) that g is constant. Thus f is a polynomial of degree m. Conversely, suppose that $f(z) = \sum_{k=0}^{m} c_k z^k$ $(c_m \neq 0)$ is a polynomial of degree m. Then

$$h(z) = z^m f(z) = c_m + c_{m-1} z + \cdots + c_0 z^m$$

is an entire function and $h(0) = c_m \neq 0$. Hence f has a pole at ∞ of order m, by Theorem 2.

(c) Immediate from (a) and (b).

References

1. L. V. Ahlfors, Complex analysis, 3rd ed., McGraw-Hill, New York, 1979.
2. R. B. Ash, Complex variables, Academic Press, New York, 1971.
3. J. B. Conway, Functions of one complex variable, Springer-Verlag, New York, 1978.
4. W. Rudin, Real and complex analysis, 2nd ed., McGraw-Hill, New York, 1974.

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