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# ERGODIC THEOREMS FOR AN ASYMPTOTICALLY NONEXPANSIVE SEMI-GROUP IN HILBERT SPACES

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## 1. Introduction

The origin of ergodic theory lies in statistical mechanics. We are interested in proving the existence of limit of time averages. The recent developements in the ergodic theory of nonlinear mappings in Hilbert space started with the result of B. Baillon([1]).

Baillon considered a nonexpansive mapping $T$ of a real Hilbert space $H$ into itself. He prove that if $T$ has fixed points in $H$ then for every $x$ in $H$, the cesàro mean: $S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T x$ converges weakly as $n \to \infty$ to a fixed point of $T$.

A corresponding theorem for a strongly continuous one parameter semi-group of nonexpansive mappings $S(t)$, $t \geq 0$ was given soon after Baillon's work by Baillon and Brezis ([2]). Also, a similar result of Baillon's work was obtained by Hirano and Takahashi for an asymptotically nonexpansive mapping ([3]).

But above results are all the cases for existence of weak limit of cesàro mean.

From the example of Genel and Lindenstrauss ([4]), it follows that there exists a nonexpansive mapping such that the time average does not converge strongly.

Therefore, Pazy ([5]) gave some further assumes on the mapping in order to assure the strong convergence of cesàro mean.

A corresponding result for a continuous one parameter semi-group of nonexpánsive mapping $S(t)$, $t\geq 0$ was proved soon after Pazy's work by J.K. Kim and K.S. Ha ([6]) and recently, J.K. Kim and K.P. Park proved the existence of strong limit of cesàro mean for an asymptotically nonexpansive mapping ([7]).

In this paper, we are going to prove the cesàro mean $A_\lambda x=\frac{1}{\lambda}\int_0^\lambda S(t)x\,dt$ converges strongly to a common fixed point $F(S)=\bigcup F(S(t))$ for an asymptotically nonexpansive semi-group $S(t)$, $t\geq 0$.

Furthermore, in the near future, we are going to study of the existence of strong limit of cesàro mean for an almost nonexpansive mapping.

## 2. Main results

Let $H$ denote a real Hilbert space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$ which is induced by inner product.

Let $C$ be a closed convex subset of $H$ and let $\{S(t):t\geq 0\}$ be a family of mapping from $C$ into itself satisfying the following conditions:

(i). $S(t+s)=S(t)S(s)$ for all $t,s\geq 0$

(ii). $S(0)x=Ix$ for all $x\in C$,

(iii). $S(t)x$ is continuous in $t\geq 0$ for all $x\in C$,

(iv). $\|S(t)x-S(t)y\|\leq \alpha_t\|x-y\|$ for all $x,y\in C$,

where $\lim_{t\to\infty}\alpha_t=1$.

The family $\{S(t): t\geq 0\}$ is called an asymptotically nonexpansive semi-group on $C$.

Let $F(S(t))$ be the set of all fixed points of $S(t)$ in $C$ for every $t \geq 0$ and $F(S) = \bigcup F(S(t))$ (common fixed point of $S(t)$, $t \geq 0$.

Let we define the cesàro mean:

$$A_\lambda x = \frac{1}{\lambda}\int_0^\lambda S(t)x dt \text{ for all } x \in C \text{ and } \lambda > 0.$$

The following theorem is well known ([3]).

THEOREM 1. ([3]) Let $C$ be a closed convex subset of a real Hilbert space $H$. And let $\{S(t) : t \geq 0\}$ be an asymptotically nonexpansive semi-group and for all $z$ in $C$, $\{S(t)z\}$ is bounded. Then the cesaro mean $\{A_\lambda x\}$ converges weakly to a common fixed point $p$ in $F(S)$.

Let $B$ be a unit ball of $l^2$ it is shown that there exists a nonexpansive mapping $T$ and a point $x$ in $B$ such that the cesàro mean does not converge strongly in $l^2$. In this paper, we will prove that one has strong convergence of $\{A_\lambda x\}$ to a common fixed point $p$ of $F(S)$ for an asymptotically uonexpansive semi-group $S(t)$, $t \geq 0$, adding suitable assumption.

PROPOSITION 2. Let $C$ and $\{S(t) : t \geq 0\}$ satisfy the same assumptions as in theorem 1. Then the $F(S)$ is nonempty and it is closed and convex subset of $C$.

PROOF. In ([3]) $F(S)$ is nonempty, closedness of $F(S)$ is nonempty, closedness of $F(S)$ is obvious. To show convexity, it is sufficient to prove that $z = (x+y)/2 \in F(S)$ for all $x, y \in F(S)$, we have

$$\|S(t)z - x\| = \|S(t)z - S(t)x\| \leq \alpha_t \|z - x\| = 1/2\alpha \|x - y\|.$$
$$\|S(t)z - y\| = \|S(t)z - S(t)y\| \leq \alpha_t \|z - y\| = 1/2\alpha \|x - y\|.$$

Since Hilbert space is uniformly convex Banach space, we have

$$\|z-S(t)x\| \leq 1/2(1-\delta(2/\alpha_t))\alpha_t\|x-y\|$$

and hence for $t\geq 0$,

$$z=\lim_{s\to\infty} S(s)z=\lim_{s\to\infty} S(s+t)z$$
$$=S(t)\lim_{s\to\infty} S(s)z=S(t)z.$$

LEMMA 3. Let $C$ and $\{S(t): t\geq 0\}$ satisfy the same assumptions as in theorem 1. Then for each $x$ in $C$ and $\varepsilon>0$, there exists $t_0>0$, such that for all $t\geq t_0$, there exists $\lambda_0>0$ satisfying

$$\|A_\lambda x-S(t)A_\lambda x\|<\varepsilon \text{ for all } \lambda\geq\lambda_0$$

PROOF. Since

$$\|A_\lambda x-u\|^2=\frac{1}{\lambda}\int_0^\lambda \|S(t)x-u\|^2dt$$
$$-\frac{1}{\lambda}\int_0^\lambda \|S(t)x-A_\lambda x\|^2dt$$

for all $x$ in $C$, $u$ in $H$.
If we set $u=S(t)A_\lambda x$, then

$$\|A_\lambda x-S(t)A_\lambda x\|^2=\frac{1}{\lambda}\int_0^\lambda \|S(s)x-S(t)A_\lambda x\|^2ds$$
$$-\frac{1}{\lambda}\int_0^\lambda \|S(s)x-A_\lambda x\|^2 ds$$

for all $t<\lambda$. Hence

$$\|A_\lambda x-S(t)A_\lambda x\|^2=\frac{1}{\lambda}\int_0^\lambda \|S(s)x-S(t)A_\lambda x\|^2 ds$$
$$+\frac{1}{\lambda}\int_t^\lambda \|S(s)x-A_\lambda x\|^2 ds$$
$$-\frac{1}{\lambda}\int_0^\lambda \|S(s)x-A_\lambda x\|^2 ds$$
$$=\frac{1}{\lambda}\int_0^{\lambda-t} \|S(s)x-S(t)A_\lambda x\|^2 ds$$

$$+\frac{1}{\lambda}\int_0^{\lambda-t}\|S(s+t)x-S(t)A_\lambda x\|^2\,ds$$
$$-\frac{1}{\lambda}\int_0^{\lambda}\|S(s)x-A_\lambda x\|^2\,ds$$
$$\leq\frac{1}{\lambda}\int_0^{t}\|S(s)x-S(t)A_\lambda x\|^2\,ds$$
$$+(\alpha_t)^2\frac{1}{\lambda}\int_0^{\lambda-t}\|S(s)x-A_\lambda x\|^2\,ds$$
$$-\frac{1}{\lambda}\int_0^{\lambda}\|S(s)x-A_\lambda x\|^2\,ds$$
$$\leq\frac{1}{\lambda}\int_0^{\lambda}\|S(s)x-S(t)A_\lambda x\|^2\,ds$$
$$+(\alpha_t{}^2-1)\frac{1}{\lambda}\int_0^{\lambda-t}\|S(s)x-A_\lambda x\|^2\,ds.$$

If we set $d$ is diameter of $\{S(s)x:\ s>0\}$, then we have $\|S(s)x-A_\lambda x\|^2\leq d^2$ for all $\lambda>0$. By the hypothesis, for all $\varepsilon>0$, there exists $t>0$ such that $(\alpha_t^2-1)\ \leq\frac{\varepsilon^2}{2d^2}$ for all $t\ >t_0$. Therefore

$$(\alpha_t^2-1)\frac{1}{\lambda}\int_0^{\lambda-t}\|S(s)t-A_\lambda x\|^2\,ds<\frac{\varepsilon^2}{2d^2}d^2=\frac{\varepsilon^2}{2}$$

and there exists $\lambda_0>0$ such that for all $\lambda>\lambda_0$

$$\frac{1}{\lambda}\int_0^{t}\|S(s)x-S(t)A_\lambda x\|^2\,ds<\frac{\varepsilon^2}{2}.$$

Hence

$$\|A_\lambda x-S(t)A_\lambda x\|<\varepsilon.$$

THEOREM 4. Let $C$ and $\{S(t):\ t\geq0\}$ satisfy the assumptions as in theorem 1. If $S(t)$ is compact for all $t\geq0$, then $A_\lambda x=\frac{1}{\lambda}\int_0^{\lambda} S(t)x\,dt$ converges strongly to a common fixed point $p$ in $F(S)$ as $\lambda\to\infty$.

PROOF. By theorem 1, $\{A_\lambda x\}$ converges weakly to a point $p$ in $F(S)$. Let $\{A_{\lambda_k}x\}$ be a subsequence of $\{A_\lambda x\}$, then since for all $z \in C$, $\{S(t)z\}$ is bounded $\{A_{\lambda_k}x\}$ is bounded. Hence, by compactness of $S(t)$, there exists a subsequence $\{A_{\lambda_{k_j}}x\}$ of $\{A_{\lambda_k}x\}$ such that $S(t)A_{\lambda_k}x$ converges strongly to a point $p_0$ in $C$. Also by lemma 2., $\|p-p_0\| \leq \lim_{j\to\infty} \|A_{\lambda_{k_j}}x - S(t)A_{\lambda_{k_j}}x\| \to 0$, uniformly in $t \geq 0$, thus $p = p_0$. Therefore, we have

$$\|A_{\lambda_{k_j}}x - p\| \leq \|A_{\lambda_{k_j}}x - S(t)A_{\lambda_{k_j}}x\| + \|S(t) \quad A_{\lambda_{k_j}}x - p\|$$

and also, it implies that $\{A_{\lambda_{k_j}}x\}$ converges strongly to $p$ in $F(S)$ for all $x \in C$.

THEOR' 5. Let $C$ and $\{S(t): \ t \geq 0\}$ satisfy the same assumptions in theorem 1. If $(I-S(t))$ transfers closed bounded subsets of $C$ into closed subsets of $H$, then for every $x \in C$, $\{A_\lambda x\}$ converges strongly to a common fixed point $p \in F(S)$ as $\lambda \to \infty$.

PROOF. We will prove that every subsequence of $\{A_\lambda x\}$ has a strongly convergent subsequence to a common fixed point of $S(t)$, $t \geq 0$. By theorem 1 $\{A_\lambda x\}$ converges weakly to a point $p \in F(S)$.

Let $\{A_{\lambda_k}x\}$ be a subsequence of $\{A_\lambda x\}$.

First case, if there exists a subsequence $\{A_{\lambda_{k_j}}x\}$ of $\{A_{\lambda_k}x\}$ such that $A_{\lambda_{k_j}}x \in F(S)$ for all integer $j$, then $\{A_{\lambda_{k_j}}x\}$ converges strongly to a point $p \in F(S)$. Second case, if there has no subsequence $\{A_{\lambda_{k_j}}x\}$ of $\{A_{\lambda_k}x\}$ with $A_{\lambda_{k_j}}x \in F(S)$ for some $j$ we can assume without loss of generality that $A_{\lambda_k}x \in F(S)$ as a subsequence. Therefore $(I-S(t))A_{\lambda_k}x \neq 0$. On the other hand, let $G=$

$\{A_{\lambda_k} x;\ k=1,2,3\cdots,\}$ (strong closure of $\{A_{\lambda_k}x\}$) then $G$ is closed and bounded set, Hence $(I-S(t))G$ is a closed set by assumption. By lemma 3, since $(I-S(t))A_{\lambda_k}x$ converges strongly to 0, uniformly on $t\geq 0$, we have,

$$0\in\overline{(I-S(t))G}=(I-S(t))G.$$

Hence there exists an element $p\in G$ such that $(I-S(t))$ $p=0$. Since $(I-S(t))A_{\lambda_k} x\neq 0$, $p\notin\{A_{\lambda_k}x\}$. Hence $p$ is an element of the derived set of $\{A_{\lambda_k}x\}$, and so there exists a subsequence $\{A_{\lambda_{k_j}} x\}$ of $\{A_{\lambda_k}x\}$ such that $\{A_{\lambda_{k_j}} x\}$ converges strongly to a point $p$ in $F(S)$. This is the complete proof of the theorem.

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