PUSAN KYŎNGNAM MATHEMATICAL JOURNAL
Vol. 1, 25~33, 1985

ON THE NUMERICAL RANGE AND HERMITIAN OPERATORS

YOUNGOH YANG

I. Introduction.

In[7], W.L. Paschke investigated right modules over a C^*-algebra B which possess a B-valued inner product respecting module action. Under this inner product, Yang defined a numerical range of an operator on this module. In this paper, which is a continuation of [8], we give analogous results of our numerical ranges as these on Banach spaces, and study its relation to spectra and various growth conditions on the resolvent. Also we define Hermitian operators in terms of our numerical range and study some results on these.

Throughout this paper we let B be a unital C^*-algebra, B^* the dual space of B, X the Hilbert B-module with a B-valued inner product $\langle\ ,\ \rangle$[7], $S(X)$ the unit sphere of X, i.e., the set of all $x \varepsilon X$ such that $\|x\|_X = \|\langle x, x\rangle\|^{\frac{1}{2}} = 1$, π_1 the projection of $X \times B^*$ onto X, $B(X)$ the set of all bounded linear operators on X, and Π the subset of $X \times B^*$ defined by

$$\Pi = \{(x, f)\ \varepsilon\ S(X) \times S(B^*):\ f(\langle x, x\rangle) = 1\}.$$

The *B-spatial numerical range* $W_B(T)$ of an operator $T \varepsilon B(X)$ is the set $W_B(T) = \{f(\langle Tx, x\rangle):\ (x, f) \varepsilon \Pi\}$[8]. This generalizes the classical concept of numerical range on a Hilbert space. The *B-spatial numerical radius* of an operator T is the number $\mathscr{W}_B(T) = \sup\{|\lambda|:\ \lambda \varepsilon\ W_B(T)\}$.

Also we denote the action of B on a right B-module X by $(x,b)\to xb(x\varepsilon X, b\varepsilon B)$. A Hilbert B-module X is assumed to have a vector space structure over the complex numbers $\mathbf{C}$ compatible with that of B in the sense that $\lambda(xb)=(\lambda x)b=x(\lambda b)$ $(x\varepsilon X,\ b\varepsilon B,\ \lambda\varepsilon \mathbf{C})$. The algebra of all bounded linear operators on X which possess bounded adjoints with respect to the B-valued inner product will be denoted by $A(X)$, and without risk of confusion, we denote the operator norm on $B(X)$ by $\|\ .\ \|$.

II. Spatial numerical ranges

LEMMA 2.1. *Let P be a subset of Π such that $\pi_1(P)$ is dense in $S(X)$ and $T\varepsilon B(X)$. Then*

(i) $\sup\{Re\ f(\langle Tx,x\rangle):\ (x,f)\varepsilon P\}=\inf\{\frac{1}{\alpha}\ (\|\ I+\alpha T\ \|-1):$

$\alpha>0\}=\lim\limits_{\alpha\to 0+}\ \frac{1}{\alpha}(\|I+\alpha T\|-1)$.

(ii) $\sup\{Re\ f(\langle Tx,x\rangle):\ (x,f)\varepsilon P\}=\sup\ \{\frac{1}{\alpha}\log\|\exp(\alpha T)\|:$

$\alpha>0\}=\lim\limits_{\alpha\to 0+}\ \frac{1}{\alpha}\ \log\|\exp(\alpha T)\|$.

(iii) $\overline{co}\{f(\langle Tx,x\rangle):\ (x,\ f)\varepsilon P\}=V(\mathrm{B}(X),T)$, *where $\overline{co}\ E$ and $V(A,a)$ denote the closed convex hull of E and the numerical range of a of a unital normed algebra A respectively.*

PROOF. (i) and (iii) See [8].

(ii) The proof is similar to that of Theorem 3.4[2].

By Theorem 2.6[2], we have $S_p(T)\subset V(B(X),\ T)=\overline{co}\ W_B(T)$ where $S_p(T)$ denotes the spectrum of T. However, the following stronger statement holds.

THEOREM 2.2. *Let $T\varepsilon B(X)$ be any operator having its* *adjoint. Then* $S_P(T)\subset\overline{W}_B(T)$; *In particular* $\partial S_P(T)\subset\overline{W}_B(T)$. ∂ *stands for "boundary of".*

PROOF. For each $T\varepsilon B(X)$, we have $S_P(T)=\Pi(T)\cup\Gamma(T)$ where $\Pi(T)$, $\Gamma(T)$ denote the approximate point spectrum, the compression spectrum of T respectively. If $\lambda\varepsilon\Gamma(T)$, then the range of $T-\lambda I$ is not dense so since $R(\lambda I-T)^{\perp}=N(\bar{\lambda}I-T^*)$, the range of $T-\lambda I$ has a nonzero orthogonal complement. Hence $\bar{\lambda}$ is an eigenvalue of T^* so that $\bar{\lambda}\ \varepsilon W_B(T^*)$ and therefore $\lambda\varepsilon W_B(T)$. On the other hand if $\lambda\varepsilon\Pi(T)$, then there exist unit vectors x_n such that $(T-\lambda I)x_n\to 0$. By the Hahn-Banach Theorem, there exists f_n in B* of unit norm so that

$f_n(\langle x_n, x_n\rangle)=1$. Then $|f_n(\langle Tx_n, x_n\rangle)-\lambda|=|f_n(\langle Tx_n-\lambda x_n, x_n\rangle)|\leqq\|Tx_n-\lambda x_n\|_X$. Thus $f_n(\langle Tx_n, x_n\rangle)\to\lambda$ as $n\to\infty$, and hence $\lambda\varepsilon\overline{W}_B(T)$. It is well known that $\partial S_P(T)\subset\Pi(T)$[6], thus in particular $\partial S_P(T)\subset\overline{W}_B(T)$.

REMARK 2.3. $\|R_\lambda\|=\|(T-\lambda I)^{-1}\|\leqq d(\lambda,\overline{W}_B(T))^{-1}$ for $\lambda\notin\overline{W}_B(T)$ $(T\varepsilon A(X))$. For given $x\varepsilon X$ with $\|x\|_X=1$, there is an f in B^* such that

$\|f\|=f(\langle x, x\rangle)=1$, and then $\|(T-\lambda I)x\|_X\geqq|f(\langle(T-\lambda I)x, x\rangle)|=|f(\langle Tx, x\rangle)-\lambda|\geqq d(\lambda,\overline{W}_B(T))$. Hence $\|(T-\lambda I)x\|_X\geqq d(\lambda,\overline{W}_B(T))\|x\|_X$ and so $\|R_\lambda\|=\|(T-\lambda I)^{-1}\|\leqq d(\lambda,\overline{W}_B(T))^{-1}$ for $\lambda\notin\overline{W}_B(T)$.

THEOREM 2.4. *If* $T\varepsilon A(X)$ *and* K *is a closed convex subset of the plane, then* $K\supset W_B(T)$ *if and only if* $\|(T-\lambda I)^{-1}\|\leqq d(\lambda, K)^{-1}$ *for* $\lambda\notin K$.

PROOF. If $K\supset W_B(T)$, then Remark implies that

$\|(T-\lambda I)^{-1}\|\leqq \mathrm{d}(\lambda,\overline{W}_B(T))^{-1}\leqq d(\lambda, K)^{-1}$ for $\lambda\notin K$.

Conversely, suppose that the resolvent of T satisfies the indicated growth condition. To show that $W_B(T)\subset K$, it suffices to show that every half-plane H which contains K also contains $W_B(T)$. By a preliminary translation and rotation, we may suppose that H is the right half-plane, $Re\ z\geqq 0$. Since $H\supset K$, $\|(I+tT)^{-1}\|=t^{-1}\|(t^{-1}I+T)^{-1}\|\leqq 1$ for all $t>0$. Hence if $(x,f)\ \varepsilon\ \Pi$, then $Re\ f(\langle(I+tT)^{-1}x,x\rangle)\leqq \|f\|\|(I+tT)^{-1}\|\ \|x\|^2{}_X\leqq 1=f(\langle x,x\rangle)$, and thus $0\leqq Re\ f(\langle(I-(I+tT)^{-1})x,x\rangle)=Re\ f(\langle tT(I+tT)^{-1}x,x\rangle)$. Dividing by t and letting $t\to 0$ yields $Re\ f(\langle Tx,x\rangle)\geqq 0$. Since (x,f) is arbitrary, this shows that $W_B(T)\subset H$.

THEOREM 2.5. *Let* $S,T\ \varepsilon\ A(X)$, $0\notin\overline{W}_B(T)$ *and*

$E=\{\lambda\mu^{-1}:\ \lambda\varepsilon\overline{W}_B(S),\ \mu\varepsilon\overline{W}_B(T)\}$. *Then* $S_P(T^{-1}S)\subset E$.

PROOF. Let z be a complex number not belonging to E. Then there exists $d>0$ such that $|z\mu-\lambda|\geqq d(\lambda\varepsilon\overline{W}_B(S)$, $\mu\varepsilon\overline{W}_B(T))$. Given $(x,f)\varepsilon\ \Pi$, we have $\|(zT-S)x\|_X\geqq|f(\langle(zT-S)x,x\rangle)|=|zf(\langle Tx,x\rangle)-f(\langle Sx,x\rangle)|\geqq d$ since $f(\langle Tx,x\rangle)\ \varepsilon\ W_B(T)$ and $f(\langle Sx,x\rangle)\ \varepsilon\ W_B(S)$. Similarly $\|(zT-S)^*x\|_X\geqq d$. By [2], we conclude that $zT-S$ is invertible. Since $0\notin\overline{W}_B(T)$ and $S_P(T)\subset\overline{W}_B(T)$, T is invertible. Therefore $zI-T^{-1}S$ is invertible, i.e., $z\notin S_P(T^{-1}S)$.

The B-numerical index of X is the real number $n_B(X)$ defined by $n_B(X)=\inf\{\mathscr{W}_B(T):\ T\ \varepsilon\ B(X),\ \|T\|=1\}$. It is obvious that $\frac{1}{e}\leqq n_B(X)\leqq 1$ by Theorem 4.1[2]. It has long been known that for a complex Hilbert space X of dimension greater than one, $n_B(X)=\frac{1}{2}$ [6].

Given $x\ \varepsilon\ S(X)$, let $D(B,\langle x,x\rangle)=\{f\ \varepsilon\ S(B^*):\ f(\langle x,x\rangle)=1\}$ and $W_B(T,x)=\{f(\langle Tx,x\rangle):\ f\ \varepsilon\ D(B,\langle x,x\rangle)\}$ $(T\ \varepsilon\ B(X))$. Then we see that $W_B(T)=\bigcup\{W_B(T,x):\ x\ \varepsilon\ S(X)\}$.

An application of the Hahn-Banach Theorem shows that for each $x \varepsilon S(X)$, we have $D(B, \langle x, x\rangle) \neq \phi$. In the weak* topology, $D(B, \langle x, x\rangle)$ is a closed convex subset of the unit ball in B^* and hence compact. Since $D(B, \langle x, x\rangle)$ is convex, $D(B, \langle x, x\rangle)$ is connected in any topology which makes B^* a topological linear space because in any such topology, $t \to tf + (1-t)g, 0 \leqq t \leqq 1$ is a continuous function. Using Lemma 15.7[3] and the fact that the sets $W_B(T, x)$ are nonvoid compact convex subsets of **C**, we obtain the following results;

LEMMA 2.6. *The mapping $x \to W_B(T, x)$ is an upper semicontinuous mapping of $S(X)$ with the norm topology into the nonvoid compact convex subset of* **C**.

From this fact, we can prove that $W_B(T)$ is connected for each $T \varepsilon B(X)$, unless X has dimension one over R.

In terms of the Hausdorff metric for compact subsets of the plane, we obtain the continuity of the function $\overline{W}_B$.

THEOREM 2.7. *The function $\overline{W}_B$ is continuous with respect to the uniform operator topology.*

PROOF. If $\|S-T\| < \varepsilon$ and $(x, f) \varepsilon \Pi$, then $|f(\langle (S-T)x, x\rangle)| \leqq \|S-T\| < \varepsilon$, and therefore $f(\langle Sx, x\rangle) = f(\langle Tx, x\rangle) + f(\langle (S-T)\ x,\ x\rangle)\ \varepsilon\ W_B(T) + (\varepsilon)$. It follows that $\overline{W}_B(S) \subset \overline{W}_B(T) + (\varepsilon)$. Similarly $\overline{W}_B(T) \subset \overline{W}_B(S) + (\varepsilon)$.

Given $T_1, \cdots, T_n \varepsilon B(X)$, we define *the B-joint numerical range of* $T = (T_1, \cdots, T_n)$ by $W(T) = \{(f(\langle T_1x, x\rangle), \cdots, f(\langle T_nx, x\rangle)) : (x, f) \varepsilon \Pi\}$. Clearly $W_B(T)$ is a bounded subset of $\mathbf{C}^n$. We say that $z = (z_1, \cdots, z_n)$ is in *the joint point spectrum* of T if there exists some nonzero element $x \varepsilon X$ such that $T_i x = z_i x (i = 1, \cdots, n)$[2]. It is obvious that the B-joint numerical range of a n-tuple T of operators includes the joint point spectrum of T.

In this paper, we have the following two problems;

(1) For what operators $T \varepsilon B(X)$ is $W_B(T)$ a closed set?

(2) Characterize those Hilbert B-modules X such that $W_B(T)$ is convex for every $T \varepsilon B(X)$.

Ⅲ. Hermitian operators

DEFINITION 3.1. An operator $T \varepsilon B(X)$ is said to be B-Hermitian if $W_B(T)$ is real. We denote by $H(X)$ the set of B-Hermitian operators of $B(X)$.

It is obvious from the definition that $H(X)$ is a real Banach space, and $i(ST-TS) \varepsilon H(X)$ if T, $S \varepsilon H(X)$. Furthermore since $\mathscr{W}_B(\cdot)$ is equivalent to $\|\cdot\|$, any operator $T \varepsilon H(X)$ for which T and iT are both B-Hermitian must be equal to zero.

THEOREM 3.2. *Let T be an operator in $B(X)$. Then the following statements are equivalent*:

(i) T is B-Hermitian.

(ii) $\|I+i\alpha T\|=1+0(\alpha)$ an $\alpha\to 0$, with α real.

(iii) $\|\exp(i\alpha T)\|=1$ for real α.

(iv) $\|\exp(i\alpha T)\|\leqq 1$ for real α.

PROOF. Since $W(\alpha+\beta T)=\alpha+\beta W_B(T)$, T is B-Hermitian if and only if $\sup Re\ W_B(iT)=\sup Re\ W_B(-iT)=0$. Hence the equivalence of (i) and (ii) follows from Lemma 2.1 (i). Also (iii) implies (i) by Lemma 2.1 (ii). If (i) holds, then again by Lemma 2.1(ii), $0=\sup\{\frac{1}{|\alpha|}\log\|\exp(i\alpha T)\|: \alpha\neq 0\}$. (*) Now $1=\|I\|\leq\|\exp(i\alpha T)\|\ \|\exp(-i\alpha T)\|$ ($\alpha \varepsilon R$). If, for some real β, we have $\|\exp(i\beta T)\|\neq 1$, the last inequality shows that there is a real number $\gamma\neq 0$ such that $\|\exp(i\ \tau\ T)\|>1$, contradicting(*). The argument just given also shows that (iii) and (iv) are equivalent. This completes the proof.

With the norm induced from $B(X)$, $A(X)$ is a C^*-algebra. It is easy to show that $T \varepsilon A(X)$ is B-Hermitian if and only if $T=T^*$.

PROPOSITION 3.3. *If $T \varepsilon B(X)$ can be written in the from $T=R+iJ$ with R and J B-Hermitian operators in $H(X)$, then R and J are uniquely determined.*

PROOF. If R' and J' are B-Hermitian operators in $B(X)$ and $T=R'+iJ'$, then $W_B(R-R')=iW_B(J'-J)=\{0\}$. Thus $R=R'$ and $J=J'$.

We let $J_B(X)=\{T+iS\colon T, S \varepsilon H(X)\}$, and we may define a mapping $*$ from $J_B(X)$ to itself by $(T+iS)^*=T-iS(T, S\varepsilon\ H(X))$. It is easy to show that $J_B(X)$ with the norm of $B(X)$ is a Banach space and $*$ is a continuous linear involution on $J_B(X)$.

DEFINITION 3.4. An operator $T \varepsilon B(X)$ is said to be B-positive if $W_B(T)\subset R^+$. We denote by $P(X)$ the set of all B-positive operators of $B(X)$.

In the real Banach space $H(X)$, the set $P(X)$ is a proper closed cone in which I is an interior point.

PROPOSITION 3.5. *Let $T \varepsilon P(X)$ and $0 \notin W_B(T)$. Then T is an interior point of $P(X)$ in $H(X)$.*

PROOF. Set $\lambda=\inf\{\alpha\colon \alpha \varepsilon W_B(T)\}$. Then $\lambda>0$ and $W_B(T+S)\subset W_B(T)+W_B(S)\subset W_B(T)+[-\|S\|, \|S\|]$ for any $S \varepsilon H(X)$. Thus if $\|S\|<\lambda$, we have $W_B(T+S)\subset R^+$, and the proposition follows.

We shall call pencil of elements of an algebra A the set of linear combinations a$-\lambda$b where $a \varepsilon A$, $b \varepsilon A$ and λ is a scalar parameter.

DEFINITION 3.6. Let T and S be elements of $B(X)$. The set $W_B^S(T)=\{\lambda: (E(x, f) \varepsilon \Pi) (f(\langle(T-\lambda S)x, x\rangle)=0)\}$ is called the *B-spatial numerical range of the pencil* $T-\lambda S$.

Under the additional assumption that $S \varepsilon H(X)$ with $S>0$, it is readily seen that

$$W_B^S(T)=\left\{\frac{f(\langle Tx,x\rangle)}{f(\langle Sx,x\rangle)} : (x,f) \varepsilon \Pi\right\}.$$

Definition 3.6 generalizes the B-spatial numerical range of an operator T since $W_B(T)=W_B^I(T)$. Generalizing B-spatial numerical radius, we also introduce

$\mathscr{W}_B^S(T)=\sup\{|\lambda|: \lambda \varepsilon W_B^S(T)\}$ and we have, in particular, $\mathscr{W}_B(T)=\mathscr{W}_B^I(T)$.

PROPOSITION 3.7. *Let* $S \varepsilon H(X)$ *and* $S>0$. *Then*

$$\mathscr{W}_B^S(T)= \inf_{\lambda>0} \{-\lambda S<T<\lambda S\} \quad (T \varepsilon B(X)).$$

PROOF. Obvious.

References

[1]. R. Beauwens & J. V. Binnebeek, *Convergence theorems in Banach algebras*, Pacific J. of Math., 68(1977), 13-24.

[2]. F. F. Bonsall & J. Duncan, *Numerical ranges of operators on normed spaces and of elements of normed algebras*, Cambridge Univ. Press, London, 1971.

[3]. ______, *Numerical ranges* II, Cambridge Univ. Press, 1973.

[4]. ______, *Complete normed algebras*, Springer-Verlag Berlin, 1973.

[5]. H. R. Dowson, *Spectral theory of linear operators*, Academic Press, London, 1978.

[6]. P. R. Halmos, *A Hilbert space problem book*, Springer-Verlag, New York, 1982.

[7]. W. L. Paschke, *Inner product modules over B^*-algebras*, Trans. Amer. Math. Soc., 182(1973). 443-468.

[8]. Y. Yang, *A note on the numerical range of an operator*, Bull. Korean Math. Soc., (1984) 27-30.

Department of Mathematics
Korea Naval Academy
Jinhae 602, Korea