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ON CS-SEMIDEVELOPABLE SPACES

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0. Instruction

In this paper cs-semidevelopable spaces are defined and shown to be the same as the semimetrizable spaces. Strongly cs-semidevelopable space are defined in a natural way and proved to coincide with an important class of semi-metric space, namely those in which "Cauchy sequence suffice". These space are shown to possess a few other interesting properties. Probably the most significant of these are that a space X is a cf-semistratifiable $w\Delta$-space if and only if it is cs-semidevelopable and that the image of a cs-semidevelopable space under a continuous pseudo open is cs-semidevelopable.

1. Cs-semidevelopable spaces

DEFINITION 1.1. (D1). A development for a space X is a sequeuce

$$\Delta = \{g_n | n \in N\}$$

of open covers of X such that $\{\mathrm{st}(x, g\ )|\ n \in N\}$ is a local base at x, for each $x \in X$. A space is developable if and only if there exists a development for the space,

DEFINITION 1.2. Let $\Delta = \{g_n|\ n \in N\}$ be a sequence of (not necessarily open) convers of space X,

(D2). Δ is a semidevelopment for X if and only if, for each $x \in X$, $\{st(x, g_n) n \in N\}$ is a local system of neighborhoods at x.

(D3). A semidevelopment of X is a strong-semidevelopment if and only if for each $M \subset X$ and $x \in \bar{M}$ there exists a descending sequence $\{G_n | n \in N\}$ such that $x \in G_n \in g_n$ and $G_n \cap M \neq \phi$.

(D4). A semidevelopment Δ for X is a point-finite semidevelopment if and only if for each $x \in X$ and for each positive integer n, x is contained in only a finite number of sets in g_n.

(D5). A semidevelopment Δ for X is a cs-semidevelopment if and only if for each convergent sequence $x_n \to x$ and for each open subset U containing $x \in X$, there is a positive integer k such that $x \in \operatorname{st}(x, g_k) \subset U$ and $\langle x_n \rangle$ is eventually in $\operatorname{st}(x, g_k)$.

A space is called semidevelopable if and only if there exists a semidevelopment for X. Similarly, X is called strongly (and/or point finite) semidevelopable if there exists a strong (and/or point-finite) semidevelopment for X.

Finally, a space X is called cs-semidevelopable if and only if there exists a cs-semidevelopment for X, Similarly that X is called strongly (and/or point-finite) cs-semidevelopable if and only if there exists a strong (and/or point-finite) cs-semidevelopment for X.

PROPOSITION 1.3. In order that a sequence $\Delta = \{g_n | n \in N\}$ of cover of a space X be a cs-semidevelopment it is necessary and sufficient that for each $M \subset X$ and $x \in \bar{M}$ there exists a sequence $\{G_n | n \in N\}$ such that

$$x \in G_n \in g_n \text{ and } G_n \cap M \neq \phi$$

PROOF. Straightforward from Definition 1.2.

For late use, we note that every (point-finite and/or strongly) cs-semidevelopable space has a (poin-finite and/or strong) cs-semidevelopment $\{g_n | n \in N\}$ having the property that $g_{n+1} < g_n$ for each positive integer $n \in N$. Hence, whenever the existence of a cs-semidevelopment is assumed in a theorem. We may assume that it has the property mentioned above cs-semidevelopments having this property shall be called refining cs-semidevelopments.

DEFINITION 1.4. A metric on a space X is a function d:

$X \times X \to R$ (real numbers) satisfying the following conditions:

For each $x, y, z \in X$ and $\phi \neq M \subset X$,

(1) $d(x, x) = 0$

(2) $d(x, y) > 0$ if $x \neq y$

(3) $d(x, y) = d(y, x)$

(4) $d(x, z) \leq d(x, y) + d(y, z)$

(5) $x \in \bar{M}$ if and only if $d(x, M) = \inf\{d(x, m) | m \in M\} = 0$

DEFINITION 1.5. A semi-metric on a space X is a function d: $X \times X \to R$ satisfying conditions (1), (2), (3) and (5) above. By a (semi-) metric space we mean a space X together with a specific (semi-) metric on X, In this paper, whenever the (semi-) metric is not specified it will be assumed to be denoted by the letter "d", the sphere about the point x of radius "ε" will be denoted by $S(x:\varepsilon)$. Note that spheres need not be open that $x \in$ Int $S(x:\varepsilon)$ if $\varepsilon \rangle 0$.

It should be noted that in most of our theorem the T_0 property is assumed. This is usually done to insure that a cs-semidevelopable space satisfies (2) in the previous definition which is satisfied in a semi-metric spaces.

DEFINITION 1.6. Let (X,d) be a semi-metric space. A sequence $\{x_n|n\in N\}$ in X is a Cauchy sequence if and only if for each $\varepsilon>0$ there exists an integer N_0 such that $d(x_n,x_m)\langle\varepsilon$ whenever $m,n\rangle$ N_0.

Note that because of the lack of the triangle inequality not all convergent sequences in a semi-metric space are necessarily Cauchy sequences.

2. Theorems for Cs-semidevelopable spaces

THEOREM 2.1. A space X is semi-metrizable if and only if it is a cs-semidevelopable space.

PROOF. Let $\Delta=\{g_n|n\in N\}$ be a refining cs-semidevelop ment for the cs-semidevelopable space where, without loss of generality, $g_1=\{X\}$. For $x,y\in X$, let $n(x,y)$ be the smallest integer n such that there is n_0 element of g_n containing both x and y. If n_0 such integer exists let $n(x,y)=\infty$

Define d: $X\times X\to R$ as follows. For $x,y\in X$, let $d(x,y)=2^{-n(x,y)}$, where $2^{-\infty}=0$. Then clearly, for every $x,y\in X$, $d(x,x)=0$ and $d(x,y)=d(y,x)$, Also if $x\neq y$, then, since X satisfies(D5) in the previous Definition 1.2., there is an open set U containing one of the points, say x but not the other. Then there is an integer n such that $x\in st(x,g_n)\subset U$. Then $y\in U$ implies $y\in st(x,g_n)$ which implies $y\in st(x,g_i)$ for each $i\geq n$.

It follows that $n(x,y)\leq n$ and hence $d(x,y)\geq 2^{-n}>0$.

New note that $S(x:2^{-n})=st(x,g_n)$ for each $x\in X$ and each integer n. For $y\in S(x:2^{-n})$ if and only if $d(x,y)<2^{-n}$ if and only if $n(x,y)>n$ if and only if there exists $G\in g_n$ such that $x,y\in G$ if and only if $y\in st(x,g_n)$. Now let $M \subset X$. Then $X \in \bar{M}$ if and only if $st(x, g_n) \cap M \neq \phi$ for each integer n if and only if $S(x:2^{-n}) \cap M \neq \phi$ for each integer n if and only if $d(x, M)=0$ Hence, d is a semi-metric on X.

Conversely, assume that d is a semi-metric on X.

For each positive integer n, let g_n be the collection of all sets of diameter less than $1/n$. Then for each n, $S(x:1/n)=st(x, g_n)$. For let $y \in S(x:1/n)$. Then $G=\{x, y\} \in g_n$ implies $y \in st(x, g_n)$. On the other hand, let $y \in st(x, g_n)$. Then there is $G \in g_n$ such that $x, y \in G$, and therefore, $d(x, y) \leqq \text{diam } G < 1/n$ thus, $y \in S(x:1/n)$.

Now let U be an open set containing the point x. Then there is an integer n such that $x \in \text{Int. } S(x:1/n) \subset S(x:1/n) \subset S(x_n:1/n) \subset U$. Therefore, $x \in \text{Int } st(x, g_n) \subset st(x, g_n) \subset st(x_n, g_n) \subset U$ and $\langle x_n \rangle$ is eventually in $st(x, g\ )$. Hence $\{g_n | n \in N\}$ is a cs-semidevelopment for X.

Corollary 2.2. Every cs-semidevelopable space is T_1.

Proof. Since every cs-semidevelopable space implies T_0 semi-developable and moreover T_0 semidevelopable spaces succeed T_1-space.

Theorem 2.3. In a cs-semidevelopable space the following conditions are equivalent:

(1) For each $M \subset X$ and each $x \in \bar{M}$, there exists a descending sequence of sets $\{G_n | n \in N\}$ of arbitrarily small diameters such that for each n, $x \in G_n$ and $x \in G_n \cap U \neq \phi$.

(2) For each $M \subset X$ and each $x \in \bar{M}$, there exists a Cauchy sequence in M converging to x.

(3) Every convergent sequence has a Cauchy Subsequence.

PROOF. Let d be a semi-metric on X since every cs-semidevelopable space implies a semi-metric space.

(1) implies(3). Let $S=\{x_n|n\in N\}$ be a sequence in X converging to the point $x\in X$. It $x_n=x$ for infinitely many n, then clearly we can define a Cauchy subsequence of S.

Otherwise let $M=\{x_n|n\in N\}\setminus\{x\}$. Then $x\in\bar{M}$ implies, by (1), that there is a descending sequence of sets $\{G_n|n\in N\}$ of arbitrarily small diameters such that for each n, $x\in G_n$ and $G_n\cap M\neq\phi$. We now define a subsequence of $\{x_n|n\in N\}$ inductively. Choose $x_{n_1}\in G_1\cap M$. Suppose x_{n_i} has been chosen for each $i=1,2,\ k-1$, such that $x_{n_i}\in G_i\cap M$ and $n_i>n_{i-1}$. Now observe that $G_k\cap M$ is infinite.

For suppose not: say $G_k\cap M=\{a_1,\cdots,\ a_m\}$. Then there exists $n_0>K$ such that diam $G_{n_0}<\min\{d(x,a_i)|i=1,2,\ m\}$ Clearly $a_i\notin G_{n_0}$ for each$=1,2,\cdots,\ m$. But then $M\cap G_{n_0}\subset M\cap G_k=\{a,\cdots a_m\}$
implies $M\subset G_{n_0}=\phi$, which is a contradiction.
Hence we can choose $x_{n_k}\in G_k\cap M$ such that $n_k>n_{k-1}$. Thus we have defined a subsequence $\{x_{n_k}|k\in N\}$ of S which is Cauchy. For let $\varepsilon>0$ be given. Then there is an integer N_0 such that diam $G_{N_0}<\varepsilon$. For $i,j\geq N_0$, We then have $x_{n_i}\in G_i\subset G_{N_0}$ and $x_{n_j}\ G_j\subset G_{N_0}$.

Thus $d(x_{n_i},x_{n_j})\leq$diam $G_{N_0}<\varepsilon$.

(3) implies(2): Assume $M\subset X$ and $x\in\bar{M}$. Since X is first countable there is a sequence $\{x_n|n\in N\}$ in M which converges to x.

By (3), this sequence has a Cauchy subsequence $\{x_{n_k}|k\in N\}$.

Then $\{x_{n_k}|k\in N\}$ is a Cauchy sequence in M converging to x.

(2) implies (1): Let $M \subset X$ and assume $x \in \bar{M}$. Then, by (2), there is a Cauchy sequence $\{x_n | n \in N\}$ in M which converges to x. For each n, let $G_n = \{x_i | i \geq u\} \cup \{x\}$. Then $\{G_n | n \in N\}$ is a descending sequence of sets of arbitrarily small diameters such that for each n, $x \in G_n$ and $G_n \cap M \neq \phi$.

Definition 2.4. A space X is strongly semi-metrizable if and only if a semi-metric satisfying any one of the conditions of the previous theorem can be realized on X.

Such a semi-metric is called a strong semi-metric.

Theorem 2.5. A space X is strongly semi-metrizable if and only if it is a strongly cs-semidevelopable space.

Proof. Let d be a strong semi-metric for X then, by Theorem 2,3 d satis fying condition (1). Now consider the cs-semidevelopment defined in Theorem 2.1.

By the definition of Δ_d and the fact that d satisfies the condition (1), it follows immediately that Δ_d is a strong cs-semidevelopment, Conversely, let $\Delta = \{g_n | n \in N\}$ be a refining strong cs-semidevelopment for X. Let d_Δ be the semi-metric on X as defined in Theorem 2.1. Observe that with this semi-metric, diam $G \leqq 2^{-n}$ for each $G \in g_n$ and $n \in N$. Thus it follows the definition of a strong semi-development that d_Δ satisfies condition (1) of the previous theorem and hence all of the conditions.

Definition 2.6. A space X as a $w\Delta$-space if and only if there is a sequence $\{g_n | n \in N\}$ of open cover of X such that, for each $x \in X$, if $x_n \in$ st(x, g_n) for $n \in N$ then the sequence $\langle x_n \rangle$ has a cluster point. Such a sequence of open covers is called a $w\Delta$-sequence for X.

THEOREM 2.7. A space X is a cf-semistratifiable $w\Delta$-space if and only if it is cs-semidevelopable.

PROOF. Let F be a cf-semistratification for a space X, and let $\Delta = \{g_n | n \in N\}$ is a $w\Delta$-sequence for the space X. We can take a st(x, g_n) such that st$(x, g_n) \subset A_\alpha \subset F(k, U)$, Where A_α is an element of any filterbase in X.

Since from definition of filterbase, g_{n+1} is an open refinement of g_i for all n. Thus $\{\text{st}(x, g_n) | n \in N\}$ is a local system of neighborhood at x, therefore $\{g_n | n \in N\}$ is a semidevelopment for X and moreover, there is a convergent sequence $\langle x_n \rangle$ is the space X since X is a $w\Delta$-space, there is a positive $k \in N$ such that $x \in st(x, g_n)$ and $x_n \in st(x, g_n) \subset U$, for all $n \in N$.

Hence the semidevelopable space implies a cs-semidevelopable space as desired.

Conversely, let $\{H_n | n \in N\}$ be an open covers of X, and let $\widehat{\mathscr{U}} = \{A_\alpha | \alpha \in \mathscr{A}\}$ be a convergent filter base for X. For each positive integer n, let $g_n = \{G | G = (\bigcap_{i=1}^{n} H_i) \cap (\bigcap_{i=1}^{n} A_{\alpha i})$, $H_i \in \widehat{H}_i$, $A_{\alpha i} \in \widehat{\mathscr{U}}\}$,

then $\{g_n | n \in N\}$ is a cs-semidevelopment for X. To show that $\{g_n | n \in N\}$ is a $w\Delta$-sequence with a cf-semistratification for X. We can choose a neighborhood $U(x)$ of x such that $x \in st(x, g_n) \subset U(x)$. Since $\{g_n | n \in N\}$ is a semidevelopment for X, and choose a sequence $\langle x_n \rangle$ such that for all n, $x_n \in st(x, g_n)$, then $x_n \in U(x)$ this implies that $\langle x_n \rangle$ converges to x since g_{n+1} is an open refinement of g_n for all $n \in N$. Hence there is $A_n \in g_n$ such that $x_n \in A_n$

$\subset st(x, g_n)$. Suppose the filter base $\mathscr{U}=\{A_\alpha|\alpha\in\mathscr{A}\}$ converging to x has a cluster point p such that $x\neq p$. Then clearly there is a positive integer k such that for a neighborhood V of p, $V(p)\cap st(x, g_k)=\phi$ Now for $n\geq k$, $A_\alpha \subset st(x, g_n)\subseteq st(x, g_k)$ for all $\alpha\geq\beta$, $\beta\in\mathscr{A}$ and so $A_\alpha\cap V(p) =\phi$ for all $\alpha\geq\beta$. This constracts the fact p is a clustor point of $\mathscr{U}$ Thus $\{g_n|n\in N\}$ is a cf-semistratifiable $w\Delta$-space.

Corollary 2.8. Let X be a reqular $w\Delta$-space. Then X is an α-space if and only if X is a cs-semidevelopable space

3. Mapping

Charles C. Alexander introduced the concept of pseudo map.

Definition 3.1. Let X and Y be topological spaces, Then a surjective map from X onto Y is pseudo-open if and only if for each $y\in Y$ and each open neighbor hood U of $f^{-1}(y)$ in X, $y\in \text{Int } f(U)$.

Theorem 3.2. The image of a cs-semidevelopable space under a continuous pseudo-open map is cs-semideleopable.

Proof. Let f be a continuous pseudo-open map from a cs-semidevelopable space X onto a space Y and $\Delta=\{g_n| n\in N\}$ a cs-semidevelopment for X. For each open V_n containing a point of Y and for all n, we can put

$$f^{-1}(V_n)=st(x, g_n).$$

Since Δ is a cs-semidevelopment for X and f is continuans, let U be any open set in X including $f^{-1}(V_n)$, then there is an convergent sequence $\langle x_n \rangle$ converging a point x belonging to $f^{-1}(y)$ in $\widehat{U}$, where $\langle y_n \rangle$ converges to y in Y. On the other hand. by Definition 1.2. there exists a $n_0 \in N$ such that $st(x, g_n)$ is contained in for all $n > n_0$ and $\langle x_n \rangle$ is eventually in $st(x, g_{n_0})$. That is, $y \in f(st(x, g_n)) \subset \text{Int} f(st(x, g_{n_0}))$ and therefore g_n is contained in Int $f(st(x, g_{n_0}))$ for all $n > n_0$.

References

1. Chanrles C. Alexander, Semi-developable spaces and Quotient images of metric spaces, Pac. J. Math. 37. No. 2, 277-293(1971).
2. R. H. Bing, Metrization of topological spaces, Conadian J, Math., 3 (1951), 175-186.
3. J. G. Ceder. Some generalizations of metric. Pak J. Math., 11 (1961) 105-125.
4. R. W. Heath, A regular semi-metric space for which there is no semi-metric under which all spheres are open, Proc. Amer. Math. Soc., 12 (1961), 810-811.
5. R. E. Hodel. Moore spaces and $w\Delta$-spaces, Pac. J. Math., 38 (1971), 641-652.
6. J. L. Kelley, General Topology, Van Nostrand, Prinston, N. J., 1955.
7. D. J. Lutzer, Semimetrizable and semistratifiable spaces, Gen, Top Appl., (1971) 42-48.
8. H. W. Martin, Metrizabibitity of M-spaces, Can. J. Math., 49, 759-763 (1972).

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