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S-CLOSEDNESS IN BITOPOLOGICAL SPACES

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1. Introduction

Let S be a subset of a topological space (X, τ). S is said to be *semi-open* [3], if for some open set U, $U \subset S \subset \mathrm{cl} U$. S is an *α-set* [5] (resp. preopen [4]), if $S \subset \mathrm{int}\,(\mathrm{cl}\,(\mathrm{int}(S)))$ (resp. $S \subset \mathrm{int}(\mathrm{cl}\,(S))$). Obviously, every α-set is semi-open and preopen. The family of all semi-open sets (resp. α-sets, preopen sets) in (X, τ) is denoted by $SO(\tau)$ (resp. $\alpha(\tau)$, $PO(\tau)$). A topological space (X, τ) is called S-closed [10] if for every semi-open cover $\{U_\alpha : \alpha \in \nabla\}$ of X there exists a finite subset ∇_0 of ∇ such that $X = \cup \{\mathrm{cl}(U_\alpha) : \alpha \in \nabla_0\}$. A subset S of a space X is regular open resp. regular closed) if $\mathrm{int}\,(\mathrm{cl}\,(S)) = S$ (resp. $\mathrm{cl}\,(\mathrm{int}(S)) = S$). Let S be a subset of a bitopological space (X, τ_1, τ_2). The closure and the interior of S with respect to τ_i are denoted by τ_i-$\mathrm{cl}(S)$ and τ_i-$\mathrm{int}(S)$, respectively, $i=1, 2$. However, τ_i-$\mathrm{cl}(S)$ (resp. τ_i-$\mathrm{int}(S)$) will be denoted by $\mathrm{cl}(S)$(resp. $\mathrm{int}(S)$) for the simplicity if the meaning is explicit.

The following results are established by T. Noiri.

THEOREM A. [7] *An open set G of a topological space X is S-closed iff it is S-closed relative to X.*

THEOREM B. [8] *An α-set A of a topological space X is an S-closed subspace iff it is S-closed relative to X.*

THEOREM C. [8] *Let A and X_0 be open sets of a space X such that $A \subset X_0$. Then A is an S-closed subspace of X_0 iff A is an S-closed subspace of X.*

In this paper, we introduce and study S-closedness in bitopological spaces. Also, we generalize the above results and introduce the generalization forms in bitopological spaces.

Throughout the paper, ∇ will stand for an index set and ∇_0, a finite subset of ∇.

2. S-closed bitopological spaces

DEFINITION 2.1. A family $\mathscr{F}$ of subsets of a bitopological space (X, τ_1, τ_2)

is called $\tau_1\tau_2$*-semi-open*, if $\mathscr{F}\subset SO(\tau_1)\cup SO(\tau_2)$. If, in addition, $\mathscr{F}$ contains at least one nonempty member of $SO(\tau_1)$ and at least one nonempty member of $SO(\tau_2)$, it is called *a pairwise semi-open family.*

A $\tau_1\tau_2$-semi-open cover, a pairwise semi-open cover, a $\tau_1\tau_2$-regular closed cover and a pairwise regular closed cover are defined similarly.

DEFINITION 2.2. A bitopological space (X, τ_1, τ_2) is said to be *pairwise S-closed* if for every pairwise semi-open cover $\{U_\alpha : \alpha\in\nabla\}$ of X, we have $X\subset \cup\{cl(U_\alpha) : \alpha\in\nabla_0\}$. If for every $\tau_1\tau_2$-semi-open cover $\{U_\alpha : \alpha\in\nabla\}$, $X\subset\cup\{cl(U_\alpha) : \alpha\in\nabla_0\}$, then (X, τ_1, τ_2) is called *M-pairwise S-closed.*

DEFINITION 2.3. [1] In a bitopological space (X, τ_1, τ_2) τ_1 *is regular with respect to* τ_2, if for every point $x\in X$ and each τ_1-closed set F such that $x\notin F$ there exist a τ_1-open set U and a τ_2-open set V such that $x\in U$, $F\subset V$ and $U\cap V=\phi$. The space is *pairwise regular* if τ_1 is regular with respect to τ_2 and τ_2 is regular with respect to τ_1.

EXAMPLE 2.1. Let X be an infinite set, τ_1=the discrete topology and $\tau_2=\{X, U\subset X : p\notin U$ for a fixed point $p\in X\}$. Then (X, τ_1, τ_2) is not pairwise S-closed for, let $\{\{x\} : x\in X, x\neq p\}\subset\tau_2$ and $\{\{p\}\}\subset\tau_1$. Then $\{\{x\} : x\in X, x\neq p\}\cup\{\{p\}\}$ is pairwise semi-open cover and there is no finite subfamily the closures of whose members cover X. Since τ_2-$cl\{x\}=\{x\}$, for every $x\in X$ and τ_1-$cl\{p\}=\{p\}$.

EXAMPLE 2.2. Let X be an infinite set, τ_1=the discrete topology and $\tau_2=\{\phi, U\subset X : p\in U$, for a fixed point $p\in X\}$. Then (X, τ_1, τ_2) is pairwise S-closed but not pairwise compact for, $\{\{x\} : x\in X\}$ is a pairwise open cover which has no finite subcover.

EXAMPLE 2.2 shows that a pairwise S-closed space may not be pairwise compact. The following theorem proves that a pairwise S-closed pairwise regular space is pairwise compact.

THEOREM 2.1. *A pairwise S-closed pairwise regular space* (X, τ_1, τ_2) *is pairwise compact.*

PROOF. Let $\{U_\alpha : \alpha\in\nabla\}$ be a pairwise open cover of X. Since X is pairwise regular, then for every τ_i-open set U_α and $x\in U_\alpha$ there exists a τ_i-open set W_α

such that $x \in W_\alpha \subset \tau_j\text{-cl}W_\alpha \subset U_\alpha$, for every $\alpha \in \nabla$; $i, j=1, 2$; $i \neq j$. Then $\{W_\alpha : \alpha \in \nabla\}$ is pairwise semi-open cover of X. Since X is pairwise S-closed, then $X \subset \bigcup \{\text{cl}(W_\alpha) : \alpha \in \nabla_0\} \subset \bigcup \{U_\alpha : \alpha \in \nabla_0\}$. This shows that X is pairwise compact.

In [2], Kim proves that every pairwise compact pairwise regular space is pairwise normal. Now, from Theorem 2.1, we can state the following corollary.

COROLLARY 2.1. *A pairwise S-closed pairwise regular space is pairwise normal.*

THEOREM 2.2. *A bitopological space (X, τ_1, τ_2) is pairwise S-closed iff every pairwise regular closed cover of X has a finite subcover.*

PROOF. Follows from the fact that every regular closed set is semi-open.

THEOREM 2.3. *A bitopological space (X, τ_1, τ_2) is M-pairwise S-closed iff it is pairwise S-closed, (X, τ_1) is S-closed and (X, τ_2) is S-closed.*

PROOF. *Necessity.* Let $\mathscr{U}$ be any one of the following three types of covers for X (i) pairwise semi-open (ii) τ_1-semi-open (iii) τ_2-semi-open. In each case, $\mathscr{U}$ is a $\tau_1\tau_2$-semi-open cover of X. Then, $X \subset \bigcup \{\text{cl}(U_\alpha) : \alpha \in \nabla_0\}$. Hence, (X, τ_1, τ_2) is pairwise S-closed, (X, τ_1) is S-closed.

Sufficiency. Let $\mathscr{U} = \{U_\alpha : \alpha \in \nabla\}$ be a $\tau_1\tau_2$-semi-open cover of X. Then $\mathscr{U}$ is pairwise semi-open or τ_1-semi-open or τ_2-semi-open. In each case, $X \subset \bigcup \{\text{cl}(U_\alpha) : \alpha \in \nabla_0\}$ and hence (X, τ_1, τ_2) is M-pairwise S-closed.

COROLLARY 2.2. *A bitopological space (X, τ_1, τ_2) is M-pairwise S-closed iff every $\tau_1\tau_2$-regular closed cover of X has a finite subcover.*

3. Remarks on S-closed subspaces

Let A be a subset of a topological space (X, τ). The relative topology induced on τ by A is denoted by τ/A and $(A, \tau/A)$ indicates the subspace of (X, τ).

If $(A, \tau/A)$ is a subspace of (X, τ), then by $\text{cl}_A(\ )$ we means the closure with respect to τ/A.

LEMMA 3.1. [6] *If A is a preopen set in X, then $cl(V) \cap A \subset \text{cl}(V \cap A)$, for every semi-open set V in X.*

LEMMA 3.2. [6] *If A and B are subsets of X such that $A \subset B$ and A preopen in X, then A is preopen in B.*

LEMMA 3.3. [6] *If B is a preopen set in X, then $X \cap B$ is semi-open in B,* *for every semi-open set V in X.*

As generalization of Theorems A, B and C we have the following results.

THEOREM 3.1. *A preopen set A of a topological space (X, τ) is an S-closed subspace iff it is S-closed relative to X.*

PROOF. *Necessity.* Let $\{V_\alpha : \alpha\in\nabla\}$ be a cover of A by semi-open sets of X. Since A is preopen in X, by Lemma 3.3 $V_\alpha \cap A$ is semi-open in A for each $\alpha\in\nabla$. Since A is S-closed, then $A=\bigcup\{\mathrm{cl}_A(V_\alpha\cap A) : \alpha\in\nabla_0\}$. Therefore, we have $A\subset\bigcup\{\mathrm{cl}(V_\alpha) : \alpha\in\nabla_0\}$. This shows that A is S-closed relative to X.

Sufficiency. Let $\{U_\alpha : \alpha\in\nabla\}$ be a cover of A by semi-open sets of the subspace A. For each $\alpha\in\nabla$, there exists a semi-open set V_α in X such that $V_\alpha\cap A=U_\alpha$ [9, Theorem 3.2]. Since $A\subset\bigcup\{V_\alpha : \alpha\in\nabla\}$ and A is S-closed relative to X, then $A\subset\bigcup\{\mathrm{cl}V_\alpha : \alpha\in\nabla_0\}$. Since A is preopen in X and by making use of Lemma 3.1, we obtain $\mathrm{cl}(V_\alpha\cap A)\subset\mathrm{cl}_A(U_\alpha)$ and hence $A=\bigcup\{\mathrm{cl}_A(U_\alpha) : \alpha\in\nabla_0\}$. This shows that A is an S-closed subspace of X.

COROLLARY 3.1. *An α-set A of a topological space (X, τ) is an S-closed subspace iff it is S-closed relative to X.*

THEOREM 3.2. *Let A and B be subsets of X such that $A\subset B$ and B preopen in X. Then A is S-closed relative to the subspace B iff A is S-closed relative to X.*

PROOF. *Necessity.* Let $\{V_\alpha : \alpha\in\nabla\}$ be a cover of A by semi-open sets of X. By Lemma 3.3, $V_\alpha\cap B$ is semi-open in B for each $\alpha\in\nabla$ and $A\subset\bigcup\{V_\alpha\cap B : \alpha\in\nabla\}$. Since A is S-closed relative to B, then, $A\subset\bigcup\{\mathrm{cl}_B(V_\alpha\cap B) : \alpha\in\nabla_0\}$. Therefore, we have $A\subset\bigcup\{\mathrm{cl}(V_\alpha) : \alpha\in\nabla_0\}$. This shows that A is S-closed relative to X.

Sufficiency. Let $\{U_\alpha : \alpha\in\nabla\}$ be a cover of A by semi-open sets of the subspace B. For each $\alpha\in\nabla$, there exists a semi-open set V_α in X such that $U_\alpha=V_\alpha\cap B$ [9, Theorem 3.2]. Since $A\subset\bigcup\{V_\alpha : \alpha\in\nabla\}$ and A is S-closed relative to X, then $A\subset\bigcup\{\mathrm{cl}V_\alpha : \alpha\in\nabla_0\}$. By making use of Lemma 3.1, we obtain $\mathrm{cl}(V_\alpha)\cap B\subset\mathrm{cl}_B(U_\alpha)$ and hence $A\subset\bigcup\{\mathrm{cl}_B U_\alpha : \alpha\in\nabla_0\}$. This shows that A is S-closed relative to B.

COROLLARY 3.2. *Let A and B be preopen subsets of X such that $A\subset B$. Then A is an S-closed subspace of B iff A is an S-closed subspace of X.*

PROOF. Follows from Lemma 3.2, Theorem 3.1 and Theorem 3.2.

4. S-closed subspaces in bitopological spaces

DEFINITION 4.1. A subset S of a bitopological space (X, τ_1, τ_2) is *pairwise S-closed relative to* X if for every pairwise semi-open (in X) cover $\{U_\alpha : \alpha \in \nabla\}$ of S, we have $S \subset \bigcup \{\mathrm{cl}(U_\alpha) : \alpha \in \nabla_0\}$.

THEOREM 4.1. *A τ_i-regular open subset of a pairwise S-closed space (X, τ_1, τ_2) is τ_j-S-closed, $(i, j=1, 2,\ i \neq j)$.*

PROOF. Let K be a proper τ_i-regular open subset of X and $\{U_\alpha : \alpha \in \nabla\}$ be a τ_j-semi-open cover of K. Then $\{U_\alpha : \alpha \in \nabla\} \cup X \backslash K$ is a pairwise semi-open cover of X. Then $X = \bigcup \{\mathrm{cl}(U_\alpha) : \alpha \in \nabla_0\} \cup (X \backslash K)$. This implies $K \subset \bigcup \{\mathrm{cl}(U^\alpha) : \alpha \in \nabla_0\}$. Therefore, K is τ_j-S-closed.

THEOREM 4.2. *Let A and B be two subsets of X such that A is pairwise S-closed relative to X and B τ_i-regular open. Then $A \cap B$ is τ_j-S-closed relative to X, $(i, j=1, 2,\ i \neq j)$.*

PROOF. Let $\{U_\alpha : \alpha \in \nabla\}$ be a τ_j-semi-open cover of $A \cap B$. Since $X \backslash B$ is τ_i-regular closed then $X \backslash B$ is τ_i-semi-open and $A \subset [\bigcup \{U_\alpha : \alpha \in \nabla\}] \cup (X \backslash B)$. Since A is pairwise S-closed relative to X, then $A \subset [\bigcup \{\mathrm{cl}(U_\alpha) : \alpha \in \nabla_0\}] \cup X \backslash B$. Therefore, $A \cap B \subset \bigcup \{\mathrm{cl}(U_\alpha) : \alpha \in \nabla_0\}$ and hence the theorem.

Let (X, τ_1, τ_2) be a bitopological space, $A \subset X$, we say that A is pairwise open (resp. preopen, semi-open, etc.) if A is open (resp. preopen, semi-open, etc.) with respect to both τ_1 and τ_2.

From the generalizations of Theorems A, B and C we have the following results in bitopological spaces.

THEOREM 4.3. *Let A be a pairwise preopen set in the bitopological space (X, τ_1, τ_2). Then A is a pairwise S-closed subspace iff A is pairwise S-closed relative to (X, τ_1, τ_2).*

PROOF. *Necessity.* Let $\{U_\alpha : \alpha \in \nabla\}$ be a pairwise semi-open (in X) cover of A. Since A is pairwise preopen, then by Lemma 3.2 $\{A \cap U_\alpha : \alpha \in \nabla\}$ is a pairwise semi-open (in A) cover of A. Since $(A, \tau_1/A, \tau_2/A)$ is pairwise S-closed, then $A \subset \bigcup \{\mathrm{cl}_A(U_\alpha \cap A) : \alpha \in \nabla_0\}$. Therefore, we have $A \subset \bigcup \{\mathrm{cl}(U_\alpha) : \alpha \in \nabla_0\}$.

Hence A is pairwise S-closed relative to X.

Sufficiency. Let $\{U_\alpha : \alpha\in\nabla\}$ be a pairwise semi-open (in A) cover of A. For each τ_i/A-semi-open set U_α, there exists a τ_i-semi-open set V_α such that $V_\alpha \cap A = U_\alpha$ [9 Theorem 3.2], $i=1, 2$. Then $A\subset\bigcup\{V_\alpha : \alpha\in\nabla\}$. Since A is pairwise S-closed relative to X, then $A\subset\bigcup\{\mathrm{cl}V_\alpha : \alpha\in\nabla_0\}$. By Lemma 3.1, $\mathrm{cl}(V_\alpha)\cap A\subset \mathrm{cl}_A(U_\alpha)$ and hence $A=\bigcup\{\mathrm{cl}_A(U_\alpha) : \alpha\in\nabla_0\}$. Then $(A, \tau_1/A, \tau_2/A)$ is pairwise S-closed.

COROLLARY 4.1. *Let A be a pairwise α-set of a bitopological space (X, τ_1, τ_2). Then, A is a pairwise S-closed subspace of X iff A is pairwise S-closed relative to X.*

THEOREM 4.4. *Let B be a pairwise preopen subset of the bitopological space (X, τ_1, τ_2), and $A\subset B$. Then A is pairwise S-closed relative to B iff A is pairwise S-close relative to X.*

PROOF. *Necessity*. Let $\{U_\alpha : \alpha\in\nabla\}$ be a pairwise semi-open (in X) cover of A. By Lemma 3.3 $\{B\cap U_\alpha : \alpha\in\nabla\}$ is a pairwise semi-open (in B) cover of A. Since A is pairwise S-closed relative to B, then $A\subset\bigcup\{\mathrm{cl}_B(U_\alpha\cap B) : \alpha\in\nabla_0\}$. Therefore, $A\subset\bigcup\{\mathrm{cl}(U_\alpha) : \alpha\in\nabla_0\}$. Hence, A is pairwise S-closed relative to B.

Sufficiency. Let $\{U_\alpha : \alpha\in\nabla\}$ be a paiwise semi-open (in B) cover of A. For each τ_i/B-semi-open set U , there exists a τ_i-semi-open set V_α such that $U_\alpha = V_\alpha\cap B$, $\alpha\in\nabla$ and $i=1, 2$. Since $A\subset\bigcup\{V_\alpha : \alpha\in\nabla_0\}$ and A is pairwise S-closed relative to X, then $A\subset\bigcup\{\mathrm{cl}(V_\alpha) : \alpha\in\nabla_0\}$. By Lemma 3.1, $\mathrm{cl}(V_\alpha)\cap B\subset \mathrm{cl}_B(U_\alpha)$ and therefore, $A\subset\bigcup\{\mathrm{cl}_B(U_\alpha) : \alpha\in V_0)$. Hence, A is pairwise S-closed relative to B.

COROLLARY 4.2. *Let A and B be pairwise preopen subsets of a bitopological space (X, τ_1, τ_2) such that $A\subset B$. Then A is a pairwise S-closed subspace of B iff A is a pairwise S-closed subspace of X.*

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