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ON THE ERROR ANALYSIS OF SOME PIECEWISE CUBIC INTERPOLATING POLYNOMIALS*

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1. Introduction

The studies on the interpolating polynomials $p(t)$, which interpolate real function $f(t)$ at the evenly spacing given knots;

$$\pi : a=t_0<t_1<\cdots\cdots<t_n=b$$

with the constraints;

(1) $$p(t_i)=f(t_i), \quad (i=0,\ 1, \cdots\cdots n),$$

(2) $$p'(t_i)=f'(t_i), \quad (i=0,\ 1, \cdots\cdots n),$$

(3) $p(t)$ is twice differentiable,

are very interesting to applied mathematician. From the Holladay's minimum curvature property theorem [4] such polynomials are called *cubic interpolating polynomials.*

By the definition of good approximation, the interpolating polynomial $p(t)$ with these constraints must be provided $\|f(t)-p(t)\|$ for a sufficiently small $\varepsilon>0$, where $\|\cdot\|$ is the Tchebycheff norm. However the graphs of such a interpolating polynomial $p(t)$ diverge even if ε is small enough and force to occur so called the Runge-Méray phenomena which oscilate at the ends of knots with large vibration [6]. It has been known that the piecewise interpolating polynomials can be avoided this phenomena.

In this paper, we derive the error bounds $\|f(t)-p(t)\|$ for three major piecewise interpolating polynomials $p(t)$, i.e. the piecewise cubic Lagrange polynomial the piecewise cubic Hermite polynomial and the cubic B-spline which have the good approximation to $f(t)$ using the Rolle's theorem even though the error bounds for such polynomials have been derived already using divided difference method by other authors.

2. The error analysis on the piecewise cubic Lagrange polynomial

DEFINITION 2.1. Given real function $f(t)$ and four evenly spacing knots

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$t_i < t_{i+1} < t_{i+2} < t_{i+3}$ of the partition π of $[a, b]$, a polynomial $l_3(t)$ of degree 3 solving the interpolation problem

$$l_3(t_{i+k}) = f(t_{i+k}), \qquad 0 \leqq k \leqq 3, \qquad t_i \in [a, b] = [t_0, t_{3d}]$$

where d is the number of subintervals, is called *the piecewise cubic Lagrange interpolating polynomial* to $f(t)$.

It has been known that

$$l_3(t) = \sum_{k=0}^{3} f(t_{i+k}) l_{i+k}(t), \quad i = 1, \cdots\cdots d,$$

where $l_{i+k}(t)$ is Lagrangian $(0 \leqq k \leqq 3)$ exist and is unique [5].

LEMMA 2.1. *Let $f(t) \in C^4[a, b]$. Then there exist a ξ in the subinterval $[t_i, t_{i+3}]$ of $[a, b]$ such that*

$$f(t) - l_3(t) = \frac{f^{(4)}(\xi)}{4!}(t-t_i)(t-t_{i+1})(t-t_{i+2})(t-t_{i+3}).$$

PROOF. Let $\bar{t} \neq t_i, t_{i+1}, t_{i+2}, t_{i+3}$ be a point in $[t_i, t_{i+3}]$. We define the function $g(t)$ by

$$g(t) = (l_3(t) - f(t)) - \frac{w(t)}{w(\bar{t})}(l_3(\bar{t}) - f(\bar{t}))$$

where $w(t) = (t-t_i)(t-t_{i+1})(t-t_{i+2})(t-t_{i+3})$. Since $g(t)$ has five zeros at the point $\bar{t}, t_i, t_{i+1}, t_{i+2}, t_{i+3}$, there are four zeros of $g'(t)$ by Rolle's theorem. Applying the Rolle's theorem repeatedly to $g'(t)$ $g''(t)$ and $g^{(3)}(t)$, there are at least one zeros of $g^{(4)}(t)$ in $[t_i, t_{i+3}]$. Since $l_3(t)$ is a polynomial of degree 3, $l_3^{(4)}(t) = 0$. Thus we can get

$$g^{(4)}(t) = -f^{(4)}(\mathrm{t}) - \frac{4!}{w(\bar{t})}(l_3(\bar{t}) - f(\bar{t})).$$

If we set ξ to be $g^{(4)}(\xi) = 0$,

$$g^{(4)}(\xi) = -f^{(4)}(\xi) - \frac{4!}{w(\bar{t})}(l_3(\bar{t}) - f(\bar{t})) = 0.$$

Hence

$$f(t) - l_3(t) = \frac{f^{(4)}(\xi)}{4!} w(t).$$

LEMMA 2.2. *Let $h = \frac{t_{i+3} - t_i}{3}$. For $t \in [t_i, t_{i+3}]$,*

$$|w(t)| = |(t-t_i)(t-t_{i+1})(t-t_{i+2})(t-t_{i+3})| \leqq \frac{3!\ h^4}{4}.$$

PROOF. $|w(t)|$ may has the maximum value at one of the distinct three points y_1, y_2, y_3 such that $t_i<y_1<t_{i+1}<y_2<t_{i+2}<y_3<t_{i+3}$. Let $|w(y_1)|$ be a maximum value of $|w(t)|$, then

$$|w(y_1)|=|(y_1-t_i)(y_1-t_{i+1})\quad(y_1-t_{i+2})(y_1-t_{i+3})|$$
$$\leqq\frac{(t_i-t_{i+1})^2}{4}|y_1-t_{i+2}||y_1-t_{i+3}|\leqq\frac{3!\ h^4}{4}.$$

Since $|y_1-t_{i+2}|<2h$. $|y_1-t_{i+3}|<3h$ and

$$|(t-t_i)(t-t_{i+1})|=\left|\left(t-\frac{t_i+t_{i+1}}{2}\right)^2-\frac{(t_i-t_{i+1})^2}{4}\right|$$
$$\leqq\frac{(ti-t_{i+1})^2}{4}=\frac{h^2}{4}.$$

When $w(y_2)$ or $w(y_3)$ is maximum value, we have the same results.

THEOREM 2.1. *Let $f\in C^4[a, b]$, then*

$$\|f(t)-l_3(t)\|\leqq\frac{1}{16}\|f^{(4)}\|h^4,\quad \textit{where } h=t_{i+1}-t_i.$$

PROOF. By Lemma 2.1 and Lemma 2.2,

$$\|f(t)-l_3(t)\|=\frac{1}{4!}\|f^{(4)}(t)w(t)\|$$
$$\leqq\frac{1}{4!}\|f^{(4)}\|\frac{3!}{4}h^4=\frac{1}{16}\|f^{(4)}\|h^4.$$

3. Error analysis on the cubic Hermite interpolating polynomial

DEFINITION 3.1. Given real function $f(t)$ and two knots $t_i<t_{i+1}$ of the partition π of $[a, b]$, a cubic polynomial $h_3(t)$ with the constraints:

$$h_3(t_i)=f(t_i),\ h_3(t_{i+1})=f(t_{i+1})$$
$$h_3'(t_i)=f'(ti),\ h_3'(t_{i+1})=f'(t_{i+1}),\quad i=0,\ 1,\ 2,\ 3\cdots\cdots,n,$$

is called the *piecewise cubic Hermite interpolating polynomial* to $f(t)$.

It has been known that the $h_3(t)$ such that

$$h_3(t)=f(t_i)H_i(t)+f(t_{i+1})H_{i+1}(t)+f'(t_i)G_i(t)+f'(t_{i+1})G_{i+1}(t)$$

where

$$H_i(t)=\left(1-2\frac{t-t_{i+1}}{t_i-t_{i+1}}\right)\left(\frac{t-t_{i+1}}{t_i-t_{i+1}}\right)^2,\quad H_{i+1}(t)=\left(1-2\frac{t-t_i}{t_{i+1}-t_i}\right)\left(\frac{t-t_i}{t_{i+1}-t_i}\right)^2,$$

and

$$G_i(t)=(t-t_i)\Big(\frac{t-t_{i+1}}{t_i-t_{i+1}}\Big)^2, \quad G_{i+1}(t)=(t-t_{i+1})\Big(\frac{t-t_i}{t_{i+1}-t_i}\Big)^2,$$

exists and is unique [2].

LEMMA 3.1. *Let $f\in C^4[a, b]$. Then there exist ξ in $[t_i, t_{i+1}]$ such that*

$$f(t)-h_3(t)=\frac{f^{(4)}(\xi)}{4!}(t-t_i)^2(t-t_{i+1})^2.$$

PROOF. Let $E(t)=(f(t)-h_3(t))-(f(\bar{t})-h_3(\bar{t}))\frac{w(t)}{w(\bar{t})}$, where $\bar{t}\neq t_i,\ t_{i+1}$ and $w(t)=(t-t_i)^2(t-t_{i+1})^2$. Since $E(t)$ vanishes at t_i, t_{i+1} and $\bar{t}$, $E'(t)$ vanishes at four distinct points $t_i<y_1<y_2<t_{i+1}$ by Rolle's theorem. And also $E''(t)$ vanishes at three points, $E^{(3)}(t)$ at two points and $E^{(4)}(t)$ at least one point ξ in $[t_i, t_{i+1}]$ from the same reason. Since $h_3^{(4)}(t)=0$, we get

$$E^{(4)}(\bar{t})=f^{(4)}(\xi)-(f(\bar{t}))-h^3(\bar{t}))\frac{4!}{w(\bar{t})}=0.$$

Thus

$$f(\bar{t})-h_3(\bar{t})=\frac{f^{(4)}(\xi)}{4!}w(\bar{t}).$$

That is,

$$f(t)-h_3(t)=\frac{f^{(4)}(\xi)}{4!}(t-t_i)^2(t-t_{i+1})^2.$$

THEOREM 3.1. *Let $f\in C^4[a, b]$, then*

$$\|f(t)-h_3(t)\|\leqq\frac{\|f^{(4)}\|}{384}h^4 \text{ with } h=t_{i+1}-t^i.$$

PROOF. Since $|(t-t_i)(t-t_{i+1})|\leq\frac{1}{4}h^2$, $|(t-t_i)^2(t-t_{i+1})^2|\leq\frac{h^4}{16}$.

Thus

$$\|f(t)-h_3(t)\|=\frac{\|f^{(4)}\|}{4!}|(t-t_i)^2(t-t_{i+1})^2|$$

$$\leq\frac{\|f^{(4)}\|}{384}h^4.$$

4. The error analysis on the cubic B-spline

DEFINITION 4.1. Given real function $f(t)$ and two knots $t_i < t_{i+1}$ of the partition π of $[a, b]$, a polynomial $s_3(t)$ of degree 3 which is twice continously differentiable on $[t_i, t_{i+1}]$ with the constraints

$$s_3(t_i)=f(t_i),\ s_3(t_{i+1})=f(t_{i+1}),$$
$$s_3'(t_i)=f'(t_i),\ s_3'(t_{i+1})=f'(t_{i+1}),$$

and

$$s_3''(t_i)=f''(t_i),\ s_3''(t_{i+1})=f''(t_{i+1}),$$

is called the *cubic spline* interpolating $f(t)$. Especially the cubic spline $s_3(t)=\sum_{k=-1}^{n+1} a_k B_k(t)$, where $B_k(t)$'s are basis functions defined by

$$B_k(t)=\frac{1}{h_3}\begin{cases}(t-t_{k-2})^3, & if\ t_{k-2}\leqq t\leqq t_{k-1},\\ h^3+3h^2(t-t_{k-1})+3h(t-t_{k-1})^2-3(t-t_{k-1})^3, & if\ t_{k-1}\leqq t\leqq t_k,\\ h^3+3h^2(t_{k+1}-t)+3h(t_{k+1}-t)^2-3(t_{k+1}-t)^3, & if\ t_k\leqq t\leqq t_{k+1},\\ (t_{k+2}-t)^3, & if\ t_{k+1}\leqq t\leqq t_{k+2},\\ 0, & \text{elsewhere},\end{cases}$$

with $k=-1, 0, 1\cdots\cdots, n,\ n+1$ and $h=t_{i+1}-t_i$, is called the *cubic B-spline* interpolating to $f(t)$. The a_k's are determined by the interpolating constraintes.

It has been known that the cubic B-spline exist and is unique [7].

LEMMA 4.1. *If* $f\in C^4[a, b]$, *then*

$$\|s_3'(t)-f'(t)\|\leqq\frac{1}{24}\|f^{(4)}\|h^3 \quad with \quad h=t_{i+1}-t_i.$$

It has proved by Birkhoff and de Boor [1].

LEMMA 4.2. *If* $f\in C^4[a, b]$, *then*

$$\|h_3(t)-s_3(t)\|\leqq\frac{1}{96}\|f^{(4)}\|h^4 \quad with \quad h=t_{i+1}-t_i.$$

PROOF. Let $e(t)=f(t)-s_3(t)$, then the $h_3(t)-s_3(t)$ becomes the piecewise cubic Hermite interpolation of $e(t)$ [3]. Hence, for t in $[t_i, t_{i+1}]$, we can write from the definition

$$h_3(t)-s_3(t)=e(t_i)H_i(t)+e(t_{i+1})H_{i+1}(t)+e'(t_i)G_i(t)$$
$$+e'(t_{i+1})G_{i+1}(t).$$

Since $e(t_i)=f(t_i)-s_3(t_i)=0$ and $e(t_{i+1})=f(t_{i+1})-s_3(t_{i+1})=0$,

$$h_3(t)-s_3(t)=e'(t_i)G_i(t)+e'(t_{i+1})G_{i+1}(t).$$

Thus by Lemma 4.1

$$\begin{aligned}\|h_3(t)-s_3(t)\| &= \|e'(t_i)G_i(t)+e'(t_{i+1})G_{i+1}(t)\| \\ &\leqq \|e'(t_i\|\|G_i(t)\|+\|e'(t_{i+1})\|\|G_{i+1}(t)\| \\ &\leqq \frac{1}{24}\|f^{(4)}\|h^3(\|G_i(t)\|+\|G_{i+1}(t)\|).\end{aligned}$$

Since $\|G_i(t)\|$ and $\|G_{i+1}(t)\|$ is less than or equal to $\frac{h}{8}$ respectively,

$$\begin{aligned}\|h_3(t)-s_3(t)\| &\leqq \frac{1}{24}\|f^{(4)}\|h^3\left(\frac{h}{8}+\frac{h}{8}\right) \\ &\leqq \frac{1}{96}\|f^{(4)}\|h^4.\end{aligned}$$

THEOREM 4.1. *Let* $f\in C^4[a,\ b]$, *then*

$$\|f(t)-s_3(t)\|\leqq\frac{5}{384}\|f^{(4)}\|h^4 \quad with \quad h=t_{i+1}-t_i.$$

PROOF. From Theorem 3.1 and Lemma 4.2,

$$\begin{aligned}\|f(t)-s_3(t)\| &= \|(f(t)-h_3(t))+(h_3(t)-s_3(t))\| \\ &\leqq \|f(t)-h_3(t)\|+\|h_3(t)-s_3(t)\| \\ &\leqq \frac{1}{384}\|f^{(4)}\|h^4+\frac{1}{96}\|f^{(4)}\|h^4 \\ &= \frac{5}{384}\|f^{(4)}\|h^4.\end{aligned}$$

5. Conclusion

By virtue of Theorem 2.1, Theorem 3.1 and Theorem 4.1, we can conclude that the piecewise cubic Hermite $h_3(t)$ has the best accuracy to interpolate a given real function $f(t)$. However, it is desirable to use the cubic B-spline $s_3(t)$ for obtaining the smoothest curve fitting because $s_3(t)\in C^2[a,\ b]$ even though $l_3(t)\in C[a,\ b]$, $h_3(t)\in C^1[a,\ b)$.

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