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ON A LIBERA INTEGRAL OPERATOR

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1. Introduction

Let $\alpha\in[0,\ 1)$ and $\beta\in(0,\ 1]$. A function $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ regular in the unit disc $U=\{z:|z|<1\}$ is said to belong to $S^*(\alpha,\ \beta)$, the class of starlike functions of order α and type β, if and only if

$$|\{zf'(z)/f(z)-1\}/\{zf'(z)/f(z)+(1-2\alpha)\}|<\beta,\ z\in U.$$

It is well known that such functions are univalent in U. The class $S^*(\alpha)$ of starlike functions of order α is identified by $S^*(\alpha)\equiv S^*(\alpha,\ 1)$. The class $S^*(0)$ is called the class of starlike functions and is denoted by S^*.

Libera [2] showed that, if $f(z)\in S^*$, then so does the function $F(z)$ defined by

$$F(z)=\frac{2}{z}\int_0^z f(t)dt. \tag{1}$$

Subsequently, Livingston [3] considered the converse problem and proved that, if $F(z)\in S^*$, then $f(z)$ belongs to S^* in $|z|<1/2$. In this paper we improve these results of Libera and Livingston for the class of starlike functions having negative coefficients.

The technique imployed by us is intirely different from those of Libera [2] and Livingston [3]. Infact, our basic tool is the following theorem due to Gupta and Jain [1].

THEOREM A. *A function* $f(z)=z-\sum_{n=2}^{\infty}|a_n|z^n$ *belongs to* $S^*(\alpha,\ \beta)$ *if and only if*

$$\sum_{n=2}^{\infty}\{(n-1)+\beta(n+1-2\alpha)\}\ |a_n|\leq 2\beta(1-\alpha).$$

The result is sharp.

We shall frequently use the above result in particular for $\beta=1$ which is due to Silverman [6].

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2. Main results

THEOREM 1. *Let* $f(z)=z-\sum_{n=2}^{\infty}|a_n|z^n$. *If* $f(z)\in S^*(\alpha, \beta)$, *then the function* $F(z)$ *defined by* (1) *belongs to* $S^*(\rho)$, *where* $\rho=\frac{3+\beta(1+2\alpha)}{3+\beta(5-2\alpha)}$. *The result is sharp. Further, the converse need not be true.*

PROOF. Since $F(z)\in S^*(\alpha, \beta)$, Theorem A ensures that

$$\sum_{n=2}^{\infty}\left\{\frac{(n-1)+\beta(n+1-2\alpha)}{2\beta(1-\alpha)}\right\}|a_n|\leq 1. \tag{2}$$

Also, from (1) we have $F(z)=z-\sum_{n=2}^{\infty}|b_n|z^n$, where $|b_n|=\left(\frac{2}{n+1}\right)|a_n|$.
Let $F(z)\in S^*(\sigma)$, then, by Theorem A, it holds if and only if

$$\sum_{n=2}^{\infty}\left(\frac{n-\sigma}{1-\sigma}\right)|b_n|\leq 1.$$

Thus we have to find the largest value of σ so that the above inequality holds. Now this inequality holds if

$$\sum_{n=2}^{\infty}\left(\frac{n-\sigma}{1-\sigma}\right)|b_n|\leq\sum_{n=2}^{\infty}\left\{\frac{(n-1)+\beta(n+1-2\alpha)}{2\beta(1-\alpha)}\right\}|a_n|$$

or if

$$\left(\frac{n-\sigma}{1-\sigma}\right)|b_n|\leq\frac{(n-1)+\beta(n+1-2\alpha)}{2\beta(1-\alpha)}|a_n|, \textit{ for each } n=2, 3, \cdots,$$

which is equivalent to

$$\sigma\leq\frac{(n+2)\{(n-1)+\beta(n+1-2\alpha)\}-4n\beta(1-\alpha)}{(n+1)\{(n-1)+\beta(n+1-2\alpha)\}-4\beta(1-\alpha)}=\rho_n, \textit{ say}, \ (n=2, 3, \cdots).$$

It is easy to verify that ρ_n is an increasing function of n. Therefore, $\rho=\inf_{n\geq 2}\rho_n=\rho_2$ and, hence $\rho=\frac{3+\beta(1+2\alpha)}{3+\beta(5-2\alpha)}$.

To show the sharpness we take the function $f(z)$ given by $f(z)=z-\frac{2\beta(1-\alpha)}{1+\beta(3-2\alpha)}z^2$. Then

$$F(z)=z-\frac{4\beta(1-\alpha)}{3\{1+\beta(3-2\alpha)\}}z^2$$

and, therefore

$$z\frac{F'(z)}{F(z)}=\frac{3\{1+\beta(3-2\alpha)\}-8\beta(1-\alpha)z}{3\{1+\beta(3-2\alpha)\}-4\beta(1-\alpha)z}=\frac{3+\beta(1+2\alpha)}{3+\beta(5-2\alpha)}, \textit{ for } z=1.$$

Hence, the result is sharp.

We now show that the converse of the theorem need not be true. To this end we consider the function

$$F(z)=z-\left(\frac{1-\rho}{3-\rho}\right)z^3.$$

Theorem A guarantees that $F(z)\in S^*(\rho)$. But the corresponding function

$$f(z)=z-2\left(\frac{1-\rho}{3-\rho}\right)z^3$$

does not belong to $S^*(\alpha, \beta)$, since, for this $f(z)$ the coefficient inequality of Theorem A is not satisfied.

As promised in the introduction, we now state a corollary of Theorem 1 which improves the result of Libera [2, Theorem 1] for the class of starlike functions having negative coefficients.

COROLLARY 1. *Let* $f(z)=z-\sum_{n=2}^{\infty}|a_n|z^n$. *If* $f(z)\in S^*$, *then the function* $F(z)$ *defined by* (1) *belongs to* $S^*(1/2)$. *The result is sharp. The converse need not be true.*

REMARK. Recently, Mocanu et al. [5] have shown that, if $f(z)=z+\sum_{n=2}^{\infty}a_n z^n \in S^*$, then the function $F(z)$ defined by (1) belongs to $S^*(.29435)$, whereas, Miller et al, [4] have shown that $F(z)\in S^*\left(\frac{\sqrt{17}-3}{4}\right)$. The above carollary provides better estimate for the order of starlikeness of $F(z)$ when the coefficients in the Taylor expansion of $f(z)$ are negative. Moreover, our result is sharp also.

THEOREM 2. *Let* $F(z)=z-\sum_{n=2}^{\infty}|a_n|z^n$. *If* $F(z)\in S^*(\alpha, \beta)$, *then the function* $f(z)$ *defined by* (1) *belongs to* $S^*(\rho)$ *in* $|z|<r^*(\rho, \alpha, \beta)$, *where*

$$r^*(\rho,\alpha,\beta)=\inf_{n\geq 2}\left[\left(\frac{1-\rho}{n-\rho}\right)\left(\frac{(n-1)+\beta(n+1-2\alpha)}{(n+1)\beta(1-\alpha)}\right)\right]^{1/(n-1)}$$

The result is sharp.

PROOF. Since $F(z)=z-\sum_{n=2}^{\infty}|a_n|z^n$, it follows from (1) that $f(z)=z-\sum_{n=2}^{\infty}\left(\frac{n+1}{2}\right)|a_n|z^n$. In order to establish the required result it suffices to show that

$$|zf'(z)/f(z)-1|<(1-\rho) \text{ in } |z|<r^*(\rho,\alpha,\beta).$$

Now

$$|zf'(z)/f(z)-1|=\left|\frac{-\sum_{n=2}^{\infty}(n-1)\left(\frac{n+1}{2}\right)|a_n|z^{n-1}}{1-\sum_{n=2}^{\infty}\left(\frac{n+1}{2}\right)|a_n|z^{n-1}}\right| \tag{3}$$

$$\leq \frac{\sum_{n=2}^{\infty}(n-1)\left(\frac{n+1}{2}\right)|a_n|\ |z|^{n-1}}{1-\sum_{n=2}^{\infty}\left(\frac{n+1}{2}\right)|a_n||z|^{n-1}}$$

$$<(1-\rho),$$

provided

$$\sum_{n=2}^{\infty}\left(\frac{n-\rho}{1-\rho}\right)\left(\frac{n+1}{2}\right)|a_n||z|^{n-1}<1. \tag{4}$$

But, for $F(z)\in S^*(\alpha,\beta)$, Theorem A ensurss that

$$\sum_{n=2}^{\infty}\left\{\frac{(n-1)+\beta(n+1-2a)}{2\beta(1-\alpha)}\right\}|a_n|\leq 1.$$

Therefore, the inequality (4) holds if

$$\left(\frac{n-\rho}{1-\rho}\right)\left(\frac{n+1}{2}\right)|a_n||z|^{n-1}<\left\{\frac{(n-1)+\beta(n+1-2\alpha)}{2\beta(1-\alpha)}\right\}|a_n|, \text{ for each } n=2,3,\cdots,$$

or if

$$|z|<\left[\left(\frac{1-\rho}{n-\rho}\right)\left(\frac{(n-1)+\beta(n+1-2\alpha)}{(n+1)\beta(1-\alpha)}\right)\right]^{1/(n-1)}, \text{ for each } n=2,3,\cdots$$

Hence, $f(z)\in S^*(\rho)$ in $|z|<r^*(\rho,\alpha,\beta)$.

Sharphess follows if we take the function $F(z)$ given by

$$F(z)=z-\frac{2\beta(1-\alpha)}{(n-1)+\beta(n+1-2\alpha)}z^n,\ n=2,3,\cdots.$$

This completes the proof of theorem.

Since $r^*(\alpha,\alpha,1)=2/3$, we have the following corollary as an immediate consequence of Theorem 2.

COROLLARY 2. *Let* $F(z)=z-\sum_{n=2}^{\infty}|a_n|z^n$. *If* $F(z)\in S^*(\alpha)$, *then the function* $f(z)$ *defined by* (1) *belongs to* $S^*(\alpha)$ *in* $|z|<2/3$. *The result is sharp with the extremal function* $F(z)=z-\left(\frac{1-\alpha}{2-\alpha}\right)z^2$.

REMARK. It is a remarkable feature corollary 2 that the radius of the disc, in which $f(z)$ belongs to $S^*(\alpha)$, is independent of α. When $\alpha=0$, the corollary improves a result of Livingston [3, Theorem 1] for the class of starlike functions having negative coefficients.

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