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GENERIC SUBMANIFOLDS WITH COMMUTATIVE SECOND FUNDAMENTAL FORMS

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A submanifold M of a Kaehlerian manifold $\overline{M}$ is called a generic submanifold (an anti-holomorphic submanifold) if the normal space $N_P(M)$ of M at P is always mapped into the tangent space $T_P(M)$ of M under the action of the almost complex structure tensor F of the ambient manifold $\overline{M}$, that is, if $FN_P(M) \subset T_P(M)$ for all $P \in M$(see [5], [6], [8] etc). For example, any real hypersurface of a Kaehlerian manifold is a generic submanifold. It is well known that any generic submanifold M of a Kaehlerian manifold admits an f-structure [7] and the partial integrability [7] of the f-structure is equivalent to the fact that (i) the second fundamental tensors of M and f-structure tensor are all commute. Moreover, for any generic submanifold M of a complex space form with constant holomorphic sectional curvature c, the suare of the length of the derivative of the second fundamental tensors is not less then $(c^2/8)p(n-p)$ (n=dim M, p=codim M) and (ii) the equality is equivalent to (2.10) appeared in §2. In this sense Okumura [4] and Maeda [3] studied real hypersurface of complex projective spaces under the conditions (i) and (ii) respectively by using the method of Riemannian fibre bundles and proved the following theorems:

THEOREM A (Okumura [4]). $\tilde{\pi}(S^{2q+1} \times S^{2r+1})$ *((q, r) is some portion of* m-1) *are the only complete hypersurfaces of a complex projective space* $CP^{m/2}$ *satisfying the condition* (i), *where* $\tilde{\pi}$ *is the projection induced from the Hopf fibration*: $S^{2m+1} \rightarrow CP^{m/2}$.

THEOREM B (Maeda [3]). $\tilde{\pi}(S^{2q+1} \times S^{2r+1})$ *((q, r) is some portion of* m-1) *are only complete hypersurfaces of* $CP^{m/2}$ *satisfying the condition* (ii).

Recently, Ki, Kim and the present author [2] and Yano and Kon [8] developed those method of Okumura and Maeda extensively for generic submanifolds with flat normal connection and proved the following theorems:

THEOREM C (Ki, Pak and Kim [2]). *Let M be an n-dimensional complete generic submanifold of a complex projective space* $CP^{m/2}$ *with flat normal* *connection. If M satisfies the condition* (i) *and the mean curvature vector is parallel in the normal bundle, then M is of the form*

$$\tilde{\pi}(S^{p_1}(r_1)\times\cdots\cdots\times S^{p_N}(r_N)),\ p_1,\cdots\cdots,p_N \text{ are odd numbers} \geqq 1,$$

$p_1+\cdots\cdots+p_N=n+1,\ r_1^2+\cdots\cdots r_N^2=1,\ N=m-n+1.$

THEOREM D (Yano and Kon [8]). *Let M be an n-dimensioanl complete generic submanifold of $CP^{m/2}$ with flat normal connection. If the condition* (ii) *with $c=4$ is satisfied at every point of M, then M is*

$$\tilde{\pi}(S^{p_1}(r_1)\times\cdots\cdots\times S^{p_N}(r_N)),\ p_1+\cdots\cdots+p_N=n+1,\ 2\leqq N\leqq n+1,\ m=n+N-1,$$

where $p_1,.\cdots\cdots p_N$ are odd numbers and $r_1^2+\cdots\cdots+r_N^2=1$.

Particularly Ki and the present author proved

THEOREM E (Ki and Pak [1]). *Let M be a complete n-dimensional generic submanifold of a 2m-dimensional Euclidean space E^{2m} with flat normal connection. If M satisfies the condition* (i) *and the mean curvature vector is parallel in the normal bundle, then M is a sphere $S^n(r)$ of dimension n, an n-dimensional plane E^n, a pythagorean product of the form*

(1) $S^{p_1}(r_1)\times\cdots\cdots S^{p_N}(r_N),\ p_1,\cdots\cdots,p_N\geqq 1,\ p_1+\cdots\cdots+p_N=n,\ 1<N<2m-n,$

or a pythagorean product of the form

(2) $S^{p_1}(r_1)\times\cdots\times S^{P_N}(r_N)\times E^p,\ p_1,\cdots,p_N,\ p\geqq 1,\ p_1+\cdots+p_N=n,\ 1<N<2m-n.$

If M is a pythagorean product of the form (1) *or* (2), *then M is of essential codimension N.*

On the other hand, a submanifold M of a Kaehlerian manifold is called an anti-invariant (totally real) submanifold if $FT_P(M)\subset N_P(M)$ for all $P\in M$ (see [9]). For anti-invariant submanifolds with commutative second fundamental tensors, the following theorem is well known:

THEOREM F (Yano and Kon [9]). *Let M be an n-dimensional* ($n>1$) *anti-invariant submanifold of a complex space form $M^{-m/2}$ (c) and M be with parallel and commutative second fundamental tensors. If the right hand side of* (1.20) *appeared in* §1 *vanishes at every point of M, then either M is totally geodesic or $c=0$. Moreover, if M is not totally geodesic, then M is a pythagorean product of the form*

$$S^1(r_1)\times\cdots\cdots\times S^1(r_n) \text{ in a } C^{n/2} \text{ in } C^{m/2},$$

or a pythagorean product of the form

$$S^1(r_1)\times\cdots\cdots\times S^1(r_N)\times E^{n-N} \text{ in a } C^{n/2} \text{ in } C^{m/2},$$

where $1\leqq N<n$.

The purpose of the present paper is to study generic submanifold with commutative second fundamental tensors immersed in complex space forms under the conditions (i) and (ii).

1. Submanifolds of Kaehlerian manifolds

Let $\overline{M}$ be a $2m$-dimensional Kaehlerian manifold covered by a system of coordinate neighborhoods $\{\overline{U}; y^i\}$ and denote by g_{ji} components of the Hermitian metric tensor and by $F_j^{\ i}$ those of the almost complex structure tensor of $\overline{M}$, where and in the sequel the indices $i, j, k, h, l, s, t, \ldots$ run over the range $\{1, 2, \cdots\cdots, 2m\}$. Then we have by definition

$$F_h^{\ i}F_j^{\ h}=-\delta_j^i, \tag{1.1}$$

$$F_j^{\ t}F_i^{\ s}g_{ts}=g_{ji}, \tag{1.2}$$

and denoting by $\overline{\nabla}_j$ the operator of covariant differentiation with respect to g_{ji},

$$\overline{\nabla}_j F_i^{\ h}=0. \tag{1.3}$$

Let M be an n-dimensional Riemannian manifold covered by a system of coordinate neighborhoods $\{U; x^a\}$ and immersed isometrically in $\overline{M}$ by the immersion $i: M\rightarrow\overline{M}$, where and in the sequel the indices $a, b, c, d, e, \ldots$ run over the range $\{1, 2, \ldots, n\}$. In the sequel we identify $i(M)$ with M itself and represent the immersion i by

$$y^i=y^i(x^a). \tag{1.4}$$

We put

$$B_a^i=\partial_a y^i, \quad \partial_a=\partial/\partial x^a \tag{1.5}$$

and denote by C_x^i mutually orthogonal unit normal vectors to M. Then, denoting by g_{cb} the induced Riemannian metric tensor of M, we have

$$g_{cb}=B_c^jB_b^ig_{ji}$$

because the immersion is isometric, and also $g_{yx}=C_y^jC_x^ig_{ji}=\delta_{yx}$ is the metric tensor of the normal bundle of M, where and in the sequel the indices $x, y, z, w, u, v, \cdots$ run over the range $\{1, 2, \cdots, p\}$ $(p=2m-n)$.

We denote by ∇_b the operator of vander Waerden-Borttolotti covariant differentiation with respect to g_{cb}. Then the equations of Gauss and Weingarten for M are given by

$$\nabla_c B_b^i = h_{cb}^x C_x^i, \tag{1.6}$$

$$\nabla_c C_x^i = -h_c{}^b{}_x B_b^i, \tag{1.7}$$

respectively, where h_{cb}^x are the second fundamental tensors with respect to the unit normals C_x^i and $h_c{}^b{}_x = h_{ca}^y g^{ab} g_{yx}$, $(g^{ab}) = (g_{ab})^{-1}$. $(g^{yx}) = (g_{yx})^{-1}$.

Therefore, equations of Gauss, Codazzi and Ricci are respectively given by

$$K_{dcba} = K_{kjih} B_d^k B_c^j B_b^i B_a^h + h_{dax} h_{cb}^x - h_{cax} h_{db}^x, \tag{1.8}$$

$$0 = K_{kjih} B_d^k B_c^j B_b^i C_x^h - (\nabla_d h_{cb}^x - \nabla_c h_{db}^x), \tag{1.9}$$

$$K_{dcyx} = K_{kjih} B_d^k B_c^j C_y^i C_x^h + (h_{dex} h_c{}^e{}_y - h_{cex} h_d{}^e{}_y), \tag{1.10}$$

where K_{kjih} and K_{dcba} are respectively the curvature tensors of $\overline{M}$ and M, and K_{dcyx} are those of the connection induced in the normal bundle of M.

We now consider the transforms $F_j^i B_b^j$ and $F_j^i C_x^j$ of B_b^j and C_x^j by the structure tensor F_j^i. Then we can put in each coordinate neighborhood U

$$F_j^i B_b^j = f_b^a B_a^i + f_b^x C_x^i, \tag{1.11}$$

$$F_j^i C_x^j = -f_x^a B_a^i + f_x^y C_y^i. \tag{1.12}$$

On the other hand, $F_{ji} = -F_{ij}$, where $F_{ji} = F_j^h g_{hi}$, which and the above equations imply

$$f_{bx} = f_{xb}, \tag{1.13}$$

$$f_{yx} = -f_{xy}, \tag{1.14}$$

where we have put $f_{bx} = f_b^y g_{yx}$, $f_{xb} = f_x^a g_{ab}$ and $f_{yx} = f_y^z g_{zx}$.

Applying F to (1.11) and (1.12), and using (1.1) and those equations, we can easily see that

$$f_a^b f_b^c + \delta_a^c = f_a^x f_x^c, \tag{1.15}$$

$$f_a^b f_b^x + f_a^y f_y^x = 0, \quad f_x^b f_b^a + f_x^y f_y^a = 0, \tag{1.16}$$

$$f_x^z f_z^y + \delta_x^y = f_x^a f_a^y. \tag{1.17}$$

Differentiating (1.11) and (1.12) covariantly along M and using (1.3), (1.6) and (1.7), we can also verify

(1.18) $$\nabla_c f_b^{\ a} = h_c^{\ a}{}_x f_b^{\ x} - h_{cb}^{\ x} f_x^{\ a},$$

(1.19) $$\nabla_b f_a^{\ x} = h_{ba}^{\ y} f_y^{\ x} - h_{be}^{\ x} f_a^{\ e},\ \nabla_b f_x^{\ a} = h_b^{\ e}{}_x f_e^{\ a} - h_b^{\ a}{}_y f_x^{\ y},$$

(1.20) $$\nabla_b f_x^{\ y} = h_{be}^{\ y} f_x^{\ e} - h_b^{\ e}{}_x f_e^{\ y}.$$

If the ambient manifold $\overline{M}$ is of constant holomorphic sectional curvature c, then, as is well known, its curvature tensors K_{kjih} have the form

(1.21) $$K_{kjih} = \frac{c}{4}(g_{kh}g_{ji} - g_{jh}g_{ki} + f_{kh}f_{ji} - f_{jh}f_{ki} - 2f_{kj}f_{ih}).$$

Therefore, the equations (1.8), (1.9) and (1.10) of Gauss, Codazzi and Ricci are respectively given by

(1.22) $$K_{dcba} = \frac{c}{4}(g_{da}g_{cb} - g_{ca}g_{db} + f_{da}f_{cb} - f_{ca}f_{db} - 2f_{dc}f_{ba}) + h_{dax}h_{cb}{}^x - h_{cax}h_{db}{}^x,$$

(1.23) $$\nabla_d h_{cb}^{\ x} - \nabla_c h_{db}^{\ x} = \frac{c}{4}(f_d^{\ x} f_{cb} - f_c^{\ x} f_{db} - 2f_{dc} f_b^{\ x}),$$

(1.24) $$K_{dcyx} = \frac{c}{4}(f_{dx}f_{cy} - f_{cx}f_{dy} - 2f_{dc}f_{yx}) + h_{dex}h_c^{\ e}{}_y - h_{cex}h_d^{\ e}{}_y.$$

2. Generic submanifolds satisfying the condition (i) of complex space forms

Let $\overline{M}^{(n+p)/2}(c)$ be a real $(n+p)$-dimensional complex space form with constant holomorphic sectional curvature (c), and let M be an n-dimensional generic submanifold with real codimension p of $\overline{M}^{(n+p)/2}(c)$. Then, by definition, M is a submanifold such that at every point P of M

$$F(N_P(M)) \subset T_P(M).$$

Therefore, acsording to our notation a submanifold M of a Kaehlerian manifold is generic if and only if $f_y^{\ x}=0$ at each point of M. Hence, in this case, the equations (1.15) (1.17), (1.18), (1.20) and (1.22) (1.24) reduce respectively to

(2.1) $$f_a^{\ b} f_b^{\ c} + \delta_a^c = f_a^{\ x} f_x^{\ c},$$

(2.2) $$f_a^{\ b} f_b^{\ x} = 0,\ f_x^{\ b} f_b^{\ a} = 0,$$

(2.3) $$f_x^{\ a} f_a^{\ y} = \delta_x^y,$$

(2.4) $$\nabla_c f_b^{\ a} = h_c^{\ a}{}_x f_b^{\ x} - h_{cb}^{\ x} f_x^{\ a},$$

(2.5) $$\nabla_b f_a^{\ x} = -h_{be}^{\ x} f_a^{\ e},\ \nabla_b f_x^{\ a} = h_b^{\ e}{}_x f_e^{\ a},$$

(2.6) $$h_{be}^{\ y} f_x^{\ e} - h_b^{\ e}{}_x f_e^{\ y} = 0,$$

(2.7) $$K_{dcba}=\frac{c}{4}(g_{da}g_{cb}-g_{ca}g_{db}+f_{da}f_{cb}-f_{ca}f_{db}-2f_{dc}f_{ba})+h_{dax}h_{cb}^{x}-h_{cax}h_{db}^{x},$$

(2.8) $$\nabla_d h_{cb}^{x}-\nabla_c h_{db}^{x}=\frac{c}{4}(f_{d}^{x}f_{cb}-f_{c}^{x}f_{db}-2f_{dc}f_{b}^{x}),$$

(2.9) $$K_{dcyx}=\frac{c}{4}(f_{dx}f_{cy}-f_{cx}f_{dy})+h_{dex}h_{c\ y}^{e}-h_{cex}h_{d\ y}^{e}.$$

First of all we prepare

LEMMA 1 (Cf. [2] and [8]). *On an n-dimensional generic submanifold of a real $(n+p)$-dimensional complex space form $\overline{M}^{(n+p)/2}(c)$, the following inequality is valid:*

$$\|\nabla_c h_{ba}^{x}\|^2\geqq\frac{c^2}{8}p(n-p).$$

Moreover, the equality is valid if and only if

(2.10) $$\nabla_c h_{ba}^{x}=\frac{c}{4}(-f_{cb}f_{a}^{x}-f_{ca}f_{b}^{x}).$$

From now on we assume that at every point of M

$$\|\nabla_c h_{ba}^{x}\|^2=\frac{c^2}{8}p(n-p).$$

and suppose that the second fundamental tensors are commutative, that is,

(2.11) $$h_{be}^{x}h_{a\ y}^{e}=h_{ae}^{x}h_{b\ y}^{e}.$$

Then, by means of Lemma 1, we have (2.10). Differentiating (2.11) covariantly along M and substituting (2.10), we can easily find

(2.12) $$\frac{c}{4}\{-(h_{aey}f_{c}^{e})f_{b}^{x}-(h_{be}^{x}f_{c}^{e})f_{ay}+(h_{bey}f_{c}^{e})f_{a}^{x}+(h_{ae}^{x}f_{c}^{e})f_{by}\}=0.$$

Transvecting (2.12) with f_{x}^{b} and using $f_{b}^{x}f_{x}^{b}=p$, we have

(2.13) $$\frac{c}{4}\{-(p-1)h_{aey}f_{c}^{e}-(f_{x}^{b}h_{be}^{x}f_{c}^{e})f_{ay}+f_{x}^{b}h_{bey}f_{c}^{e}f_{a}^{x}\}=0,$$

from which, transvecting with f_{d}^{a} and f^{ay} respectively, we can obtain

(2.14) $$\frac{c}{4}(p-1)h_{aey}f_{d}^{a}f_{c}^{e}=0,$$

(2.15) $$\frac{c}{4}(p-1)h_{aey}f^{ay}f_{c}^{e}=0.$$

with the aid of (2.6).

We now apply the operator ∇_d to (2.10) and use the Ricci identities. We then have

$$-K_{dcb}^{\ \ \ e}h_{ea}^{x}-K_{dca}^{\ \ \ e}h_{be}^{x}+K_{dcy}^{\ \ \ x}h_{ba}^{y}+\frac{c}{4}\{(\nabla_d f_{ca}-\nabla_c f_{da})f_{b}^{x}+(\nabla_d f_{cb}-\nabla_c f_{db})f_{a}^{x}$$

$$+f_{cb}\nabla_d f^x_a - f_{db}\nabla_c f^x_a + f_{ca}\nabla_d f^x_b - f_{da}\nabla_c f^x_b\} = 0,$$

from which, substituting (2.4), (2.5), (2.7) and (2.9),

$$\begin{aligned}(2.16)\qquad & \frac{c}{4}\{-h^x_{da}g_{cb} + h^x_{ca}g_{db} - f_{cb}h^x_{ae}f^e_d + f_{db}h^x_{ae}f^e_c + 2f_{dc}h^x_{ae}f^e_b\} \\ & -h_d{}^e{}_y h^x_{ea} h^y_{cb} + h_c{}^e{}_y h^x_{ea} h^y_{db} + \frac{c}{4}\{-h^x_{db}g_{ca} + h^x_{cb}g_{da} - f_{ca}h^x_{be}f^e_d \\ & + f_{da}h^x_{be}f^e_c + 2f_{dc}h^x_{be}f^e_a\} - h_d{}^e{}_y h^x_{eb} h^y_{ca} + h_c{}^e{}_y h^x_{eb} h^y_{da} + \frac{c}{4}(f^x_d f_{cy} \\ & - f^x_c f_{dy})h^y_{ba} + \frac{c}{4}\{(h_{cay}f^y_c - h_{cay}f^y_d)f^x_b + (h_{dby}f^y_c - h_{cby}f^y_d)f^x_a \\ & - f_{ca}h^x_{de}f^e_b + f_{da}h^x_{ce}f^e_b - f_{cb}h^x_{de}f^e_a + f_{db}h^x_{ce}f^e_a\} = 0,\end{aligned}$$

where we have used the hypotheses that the second fundamental forms are commutative. If we transvect with g^{da} to (2.16), then we can obtain

$$\begin{aligned}(2.17)\qquad & \frac{c}{4}\{-h^x g_{cb} - 4h^x_{ae}f^a_c f^e_b\} - h_{dey}h^{dex}h^y_{cb} + \frac{c}{4}(n+3)h^x_{cb} - \frac{c}{4}f^y_c f^e_y h^x_{be} \\ & + h^y h_c{}^e{}_y h^x_{eb} - \frac{c}{4}\{f^x_c h_{bdy}f^{dy} - h_y f^y_c f^x_b + f^x_b h_{cdy}f^{dy} + f^y_b f^e_y h^x_{ce}\} = 0,\end{aligned}$$

h^x being the mean curvature with respect to the unit normal vector C_x and defined by $h^x = g^{cb}h^x_{cb}$, from which, taking the skew-symmetric part,

$$\frac{c}{4}\{h_y f^y_c f^x_b - h_y f^y_b f^x_c\} = 0,$$

and consequently

$$\frac{c}{4}(p-1)h^x = 0.$$

Thus we have

THEOREM 1. *Let M be an n-dimensional generic submanifold with real codimension $p>1$ of a complex space form $\overline{M}^{(n+p)/2}(c)$ $(c\neq 0)$. If the second fundamental tensors h^x_{ba} are commutative and satisfy*

$$\|\nabla_c h^x_{ba}\|^2 = \frac{c^2}{8}p(n-p)$$

at every point of M, then M is minimal.

We now come back to (2.13). Substituting (2.14) and (2.15) in (2.13) gives

$$(2.18)\qquad \frac{c}{4}(p-1)(p-2)h_{aey}f^e_c = 0.$$

Differentiating (2.18) covariantly along M and taking account of (2.4) and (2.10), we obtain

$$\frac{c}{4}(p-1)(p-2)\{-\frac{c}{4}f_{be}f_{ay}f^{e}_{\ c}+h_{aey}(h_{b\ x}^{\ e}f^{x}_{\ c}-h_{bc}^{\ x}f^{e}_{\ x})\}=0,$$

from which, transvecting with $f^{c}_{\ d}$ and using $f^{e}_{\ b}f^{a}_{\ e}f^{c}_{\ a}+f^{c}_{\ b}=0$ and (2.18), it must be that

$$\frac{c}{4}(p-1)(p-2)f_{ay}f_{bd}=0,$$

and consequently

$$\frac{c}{4}(p-1)(p-2)(p-n)=0$$

with the aid of (2.1).

Thus we have

THEOREM 2. *Let M be an n-dimensional generic submanifold of a complex space form* $\overline{M}^{(n+p)/2}(c)$. *If the second fundamental tensors* $h_{ba}^{\ x}$ *are commutative and satisfy*

$$\|\nabla_{c}h_{ba}^{\ x}\|^2=\frac{c^2}{8}p(n-p)$$

at every point of M, *then*

$$c(p-1)(p-2)(p-n)=0.$$

3. Generic submanifolds with partially integrable f-structure

As already mentioned in §2, (2.1) and (2.2) imply

$$f^{e}_{\ b}f^{d}_{\ e}f^{a}_{\ d}+f^{a}_{\ b}=0,$$

which means that the induced tensor field $f^{a}_{\ b}$ defines an f-structure of rank $n-p$. We consider a distribution L defined by

$$L_P=\{X^a\in T_p(M)\mid f^{x}_{\ a}X^a=0\}$$

at each point $P\in M$. If the distribution L is integrable and moreover if the almost complex structure induced from $f^{a}_{\ b}$ on each integral manifold of L is integrable, then the f-structure $f^{a}_{\ b}$ is said to be partially integrable (see [7], [8]). For the partially integrability of the induced f-structure $f^{a}_{\ b}$, the following theorem is well known:

LEMMA 2. (Cf. [2], [8]). *Let M be an n-dimensional generic submanifold of a Kaehlerian manifold. Then the induced f-structure* $f^{a}_{\ b}$ *is partially integrable if and only if*

(3.1) $$h_{be}^{\ x}f^{e}_{\ a}+h_{ae}^{\ x}f^{e}_{\ b}=0$$

at every point of M.

Now we assume that the f-structure f_b^a is partially integrable. Then, by means of Lemma 2, we have (3.1). We transvect with f_y^b to (3.1). Then it follows that

$$h_{be}^x f_a^e f_y^b = 0,$$

from which, transvecting with f_c^a,

$$h_{bc}^x f_y^b = P_{yz}^x f_c^z, \tag{3.2}$$

where and in the sequel $P_{yz}^x = h_{be}^x f_y^b f_z^e$. Putting $P_{yzx} = P_{yz}^w g_{wx}$, we notice that P_{yzx} are symmetric for all indices x, y, z because of (2.6).

Applying the operator ∇_d to the both sides of (3.2) and then taking the skew-symmetric part with respect to the indices d and c, we get

$$-\frac{c}{2} f_{dc}\delta_y^x + h_{cb}^x h_d^e{}_y f_e^b - h_{db}^x h_c^e{}_y f_e^b = (\nabla_d P_{yz}^x) f_c^z - (\nabla_c P_{yz}^x) f_d^z$$
$$- P_{yz}^x h_{de}^z f_c^e + P_{yz}^x h_{ce}^z f_d^e$$

with the aid of (2.3), (2.5) and (2.8). Therefore, using (3.1) and the hypothesis that the second fundamental tensors are commutative, the last equation reduces to

$$-\frac{c}{2} f_{dc}\delta_y^x - 2h_{be}^x h_d^e{}_y f_c^b = (\nabla_d P_{yz}^x) f_c^z - (\nabla_c P_{yz}^x) f_d^z - 2P_{yz}^x h_{de}^z f_c^e. \tag{3.3}$$

Transvecting with f_w^c to (3.3) and using (2.2), we find

$$\nabla_d P_{yw}^x = f_w^c (\nabla_c P_{yz}^x) f_d^z,$$

from which, taking account of $P_{yz}^x = P_{zy}^x$,

$$(\nabla_d P_{wy}^x) f_b^y = f_w (\nabla_c P_{yz}^x) f_d^z f_b^y.$$

Consequently (3.3) becomes

$$f_{dc}\delta_y^x + h_{be}^x h_d^e{}_y f_c^b = P_{yz}^x h_{de}^z f_c^e,$$

from which, transvecting with f_a^c, we have

$$\frac{c}{4}(g_{da} - f_d^z f_{za})\delta_y^x - h_{ae}^x h_d^e{}_y + h_{be}^x h_d^e{}_y f_a^z f_z^b$$
$$= -P_{yz}^x h_{da}^z + P_{yz}^x h_{de}^z f_a^w f_w^e.$$

On the other hand, a direct computation by using the commutativity of the second fundamental tensors and (3.2) imply

$$\begin{aligned} P_{yz}^{\ x} h_{de}^{\ z} f_{\ a}^{w} f_{\ w}^{e} &= h_{bc}^{\ x} f_{\ y}^{b} f_{\ z}^{c} h_{de}^{\ z} f_{\ a}^{w} f_{\ w}^{e} \\ &= h_{bc}^{\ x} f_{\ y}^{b} f_{\ z}^{c} h_{dew} f_{\ a}^{w} f^{ze} \\ &= h_{bc}^{\ x} f_{\ y}^{b} (g^{ce} + f_{\ s}^{c} f^{es}) h_{dew} f_{\ a}^{w} \\ &= h_{be}^{\ x} h_{d\ w}^{\ e} f_{\ y}^{b} f_{\ a}^{w} \\ &= h_{de}^{\ x} h_{b\ w}^{\ e} f_{\ y}^{b} f_{\ a}^{w} \\ &= h_{be}^{\ x} h_{d\ y}^{\ e} f_{\ w}^{b} f_{\ a}^{w}, \end{aligned}$$

which and the above equation yield

$$h_{ae}^{\ x} h_{d\ y}^{\ e} = P_{yz}^{\ x} h_{ad}^{\ z} + (g_{ad} - f_{\ a}^{z} f_{zd}) \frac{c}{4} \delta_{y}^{x} \tag{3.4}$$

and consequently

$$h_{ae}^{\ x} h_{\ \ x}^{ae} = P_{z} h^{z} + \frac{c}{4}(n-p)p, \tag{3.5}$$

where we have put $P_z = g^{yx} P_{yxz}$.

We next prove

LEMMA 3. *Let M be an n-dimensional generic submanifold of a complex space form $\overline{M}^{(n+p)/2}(c)$. If the induced f-structure $f_{\ b}^{a}$ is partially integrable and if the second fundamental tensors are commutative, then*

$$\nabla_c h^x = \nabla_c P^x$$

at every point of M.

PROOF. By means of Lemma 2 our assumptions imply (3.1). Applying the operator ∇_c to (3.1) and substituting (2.4), we have

$$(\nabla_c h_{be}^{\ x}) f_{\ a}^{e} + h_{be}^{\ x} (h_{c\ y}^{\ e} f_{\ a}^{y} - h_{ca}^{\ y} f_{\ y}^{e}) + (\nabla_c h_{ae}^{\ x}) f_{\ b}^{e} + h_{ae}^{\ x} (h_{c\ y}^{\ e} f_{\ b}^{y} - h_{cb}^{\ y} f_{\ y}^{e}) = 0.$$

Therefore, substituting (3.4) in the last equation and using (3.2), we can easily see that

$$(\nabla_c h_{be}^{\ x}) f_{\ a}^{e} + (\nabla_c h_{ae}^{\ x}) f_{\ b}^{e} + \frac{c}{4} \{(g_{cb} - f_{ce}^{z} f_{zb}) f_{\ a}^{x} + (g_{ca} - f_{\ c}^{z} f_{za}) f_{\ b}^{x}\} = 0,$$

from which, transvecting with $f_{\ d}^{a}$,

$$-\nabla_c h_{bd}^{\ x} + (\nabla_c h_{ae}^{\ x}) f_{\ d}^{a} f_{\ b}^{e} + (\nabla_c h_{be}^{\ x}) f_{\ d}^{z} f_{\ z}^{e} + \frac{c}{4} f_{dc} f_{\ b}^{x} = 0.$$

Transvecting with g^{cd} to this equation and using the equation (2.8) of Codazzi, we obtain

$$-\nabla_b h^x + (\nabla_e h_{ca}^{\ x} + f_c^{\ x} f_{ea} - f_e^{\ x} f_{ca} - 2f_{ce} f_a^{\ x}) f^{ca} f_b^{\ e} + (\nabla_b h_{ce}^{\ x} + f_c^{\ x} f_{be}$$
$$- f_e^{\ x} f_{ca} - 2f_{cb} f_e^{\ x}) f^{cz} f_z^{\ e} = 0,$$

which yields

$$\nabla_b h^x = (\nabla_b h_{ce}^{\ x}) f^{cz} f_z^{\ e}. \tag{3.6}$$

On the other side

$$P^x = h_{ce}^{\ x} f_z^{\ e} f^{zc},$$

and hence applying the operator ∇_b to the both sides of this equation and using (2.5), (3.2) and (3.6) yield

$$\nabla_b P^x = \nabla_b h^x + P_{zy}^{\ \ x} f_e^{\ z} h_b^{ay} f_a^{\ e} + P_{yz}^{\ \ x} f_c^{\ z} h_b^{ey} f_e^{\ c},$$

which gives

$$\nabla_b P^x = \nabla_b h^x$$

Now we compute the Laplasian ΔS of a function $S = h_{ba}^{\ \ x} h^{ba}_{\ \ x}$ globally defined on M, where $\Delta = g^{dc}\nabla_d \nabla_c$. Then we have by definition

$$\frac{1}{2}\Delta S = g^{dc}(\nabla_d \nabla_c h_{ba}^{\ \ x}) h^{ba}_{\ \ x} + \|\nabla_c h_{ba}^{\ \ x}\|^2,$$

or using(2.4), (2.5) and the Ricci identity,

$$\frac{1}{2}\Delta S = g^{dc}(\nabla_b \nabla_d h_{ca}^{\ \ x}) h^{ba}_{\ \ x} + K_b^e h_{ae}^{\ \ x} h^{ba}_{\ \ x} - K_{dba}^{\ \ \ \ e} h_e^{\ dx} h^{ba}_{\ \ x} + K_{cby}^{\ \ \ \ x} h_a^{\ cy} h^{ba}_{\ \ x}$$
$$+ \frac{c}{4}\{(\nabla_c f_b^{\ x}) f_a^{\ c} + f_b^{\ x} \nabla^c f_{ac} - 2(\nabla^c f_{cb}) f_a^{\ x} + 2 f_b^{\ c} \nabla_c f_a^{\ x}\} h^{ba}_{\ \ x} + \|\nabla_c h_{ba}^{\ \ x}\|^2,$$

where $k_b^e = g^{dc} K_{bdc}^{\ \ \ \ e}$ is the Ricci tensor of M.

Here, substituting (2.4), (2.5), (2.7) and (2.9) and using (3.1), We can easily obtain

$$\frac{1}{2}\Delta S = (\nabla_b \nabla_a h^x) h^{ba}_{\ \ x} - \frac{c}{4} h^x h_x + \frac{c}{4}(n-3) h_{ba}^{\ \ x} h^{ba}_{\ \ x} - (h_{dc}^{\ \ y} h^{dc}_{\ \ x})(h_{bay} h^{bax})$$
$$+ h_y h_{bc}^{\ \ y} h_a^{\ cx} h^{ba}_{\ \ x} + 3\frac{c}{4} h_y f_a^{\ y} f_b^{\ x} h^{ba}_{\ \ x} + \|\nabla_c h_{ba}^{\ \ x}\|^2, \tag{3.7}$$

Where we have used the hypothesis that the second fundamental tensor are commutative.

But, by using (3.4) and (3.5), (3.7) can be rewritten as follow;

$$(3.8)\qquad \frac{1}{2}\Delta S=(\nabla_b\nabla_a h^x)h^{ba}_{\ x}-\frac{c}{4}h_x h^x+\frac{c}{4}(n-3)h_{ba}^{\ \ x}h^{ba}_{\ \ x}$$
$$-\{P_{xz}^{\ \ y}h^z+\frac{c}{4}(n-p)\delta_x^y\}\{P_{yw}^{\ \ x}h^w+\frac{c}{4}(n-p)\delta_y^x\}$$
$$+h_y h_{bc}^{\ \ y}P_z h^{cbz}+\frac{c}{4}(g^{cb}-f^{cz}f_z^{\ b})p+3\frac{c}{4}h_y P^y+\|\nabla_c h_{ba}^{\ \ x}\|^2$$
$$=(\nabla_b\nabla_a h^x)h^{ba}_{\ \ x}+\frac{c}{4}(p-1)h_x h^x-P_{xz}^{\ \ y}P_{yw}^{\ \ x}h^z h^w+P^z P_{yz}^{\ \ x}h^y h_x$$
$$+\left(\frac{c}{4}\right)^2(p-1)(n-p)+\{\|\nabla_c h_{ba}^{\ \ x}\|^2-2p(n-p)\}$$

On the other hand, using (3.2) and (3.4), we can compute the following identities:

$$P_{xz}^{\ \ y}P_{yw}^{\ \ x}h^z h^w=h_{bc}^{\ \ y}f^b_{\ x}f^c_{\ z}h_{ad}^{\ \ x}f^a_{\ y}f^d_{\ w}h^z h^w=h_{bcz}f^b_{\ x}f^{cy}f^a_{\ y}h_{ad}^{\ \ x}f^d_{\ w}h^z h^w$$
$$=h_{bax}h^{ax}_{\ \ d}f^b_{\ z}f^d_{\ w}h^z h^w=P_y h_{bd}^{\ \ y}+(g_{bd}-f^y_{\ b}f_{yd})pf^b_{\ z}f^d_{\ w}h^z h^w$$
$$=P_y P_{zw}^{\ \ y}h^z h^w.$$

Hence (3.8) reduces to

$$(3.9)\qquad \frac{1}{2}\Delta S=(\nabla_b\nabla_a h^x)h^{ba}_{\ \ x}+\frac{c}{4}(p-1)h^x h_x+\left(\frac{c}{4}\right)^2p(p-1)(n-p)$$
$$+\{\|\nabla_c h_{ba}^{\ \ x}\|^2-\frac{c^2}{8}\,p(n-p)\}.$$

if the mean curvature vector h^x is parallel in the normal bundle of M, then by means of Lemma 3 $\nabla_b P^x=0$, which and (3.5) imply that $h_{ba}^{\ \ x}h^{ba}_{\ \ x}$ is constant. Thus we have from (3.9) and Lemma 1

THEOREM 4. *Let M be an n-dimensional generic submanifold of a complex space form $\overline{M}^{(n+p)/2}(c)$ $(c\geqq 0)$ with parallel mean curvature vector in the normal bundle of M. If the f-structure $f^a_{\ b}$ is partially integrable and if the second fundamental tensors are commutative, then*

$$c(p-1)(n-p)=0$$

and

$$c(p-1)h_x h^x=0,\quad \|\nabla_c h_{ba}^{\ \ x}\|^2=\frac{c^2}{8}\,p(n-p)$$

at every point of M.

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