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THE PARALLELIZABILITY OF DOLD MANIFOLDS

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1. Introduction

A smooth n-dimensional manifold M is parallelizable whenever its tangent bundle $\tau(M)$ is the trivial bundle over M, equivalently, the tangent bundle $\tau(M)$ has n linearly independent vector fields on M. M is stably parallelizable if there is a trivial bundle over M whose Whitney sum with the tangent bundle $\tau(M)$ of M is trivial. The question of parallelizability of a smooth manifold has many concern in topology and geometry. For example, any Lie group G is parallelizable: the left invariant vector fields provide a basis of vector fields $\tau(G)$. The spheres S^1, S^3, and S^7 are also parallelizable; in fact S^1 and S^3 are the underlying manifolds of the Lie group $U(1)$ and $U(2)$, respectively. Since parallelizability of the real projective space $RP(n)$ would imply parallelizability of the corresponding sphere S^n, it follows that $RP(1)$, $RP(3)$, and $RP(7)$ are the only parallelizable real projective spaces. No complex projective spaces $CP(n)$ are parallelizable $(n>1)$.

A parallelizable manifold M is clearly stably parallelizable. The problem whether a stably parallelizable manifold is actually parallelizable is reduced to the problem in algebraic topology by classical theorems of Kervaire and Adams:

Let M be a stably parallelizable manifold. If dim M is even, then M is parallelizable if and only if its Euler characteristic is zero. If dim M is 1, 3, or 7, then M is parallelizable. Finally, if M has an odd dimension different from 1, 3, and 7, then M is parallelizable if and only if the semicharacteristic $\sum_{i=0}^{[n/2]} \dim_{Z_2} H_i(M:Z_2)$ is even.

In this paper, we shall be concerned with the stably parallelizability of Dold manifolds $D(m,n)$, defined by A. Dold in his study of cobordism theory, which is regarded as a generalization of the real and complex projective space.

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2. Definitions

Let S^m, $m\geqq 0$, denote the unit m-sphere in R^{m+1} with the coordinates x_0, x_1, $\cdots, x_m$ and let $CP(n)$, $n\geqq 0$ denote the complex projective n-space with the homogeneous coordinates z_0, $z_1, \cdots, z_n$. Consider the product space $S^m\times CP(n)$ and define a homeomorphism

$$T: S^m\times CP(n)\rightarrow S^m\times CP(n)$$

by

$$T(x, z)=(-x, \bar{z}), \ x\in S^m, \ z\in CP(n),$$

where $-x$ is the antipodal point of x and $\bar{z}$ is the conjugate point of z. Then, by definition, the $D(m,n)$ of dimension $m+2n$, is the quotient space obtained from $S^m\times CP(n)$ by identifying (x, z) with $T(x,z)$. $D(m,0)$ and $D(0,n)$ are readily seen to be $RP(m)$ and $CP(n)$ respectively.

The projection $S^m\times CP(n)\rightarrow S^m$ induces naturally a map p of $D(m, n)$ onto the real projective m-space $RP(m)$, and

$$\{D(m,n), \ p, \ RP(m), \ CP(n), \ Z_2\}$$

is a fibre bundle whose fibre is $CP(n)$ and the structure group is Z_2 (conjugation is the nontrivial element of Z_2).

3. Cohomology

Let $C_i^+(C_i^-)$ denote an open i-cell of S^m defined by $x_{i+1}=x_{i+2}=\cdots=x_m=0$, $x_i>0$ $(x_i<0)$ and D_j denote an open j-cell of $CP(n)$ defined by

$$z_j=1, \ z_{j+1}=z_{j+2}=\cdots=z_n=0.$$

Then $\{C_i^{\pm}\times D_j | i=0, 1, \cdots, m; j=0,1,\cdots,n\}$ forms an oriented cellular decomposition of $S^m\times CP(n)$ whose boundary relations are given by

$$\partial(C_i^{\pm}\times D_j)=\pm(C_{i-1}^{+}\times D_j+C_{i-1}^{-}\times D_j),$$

$$\partial(C_0^{\pm}\times D_j)=0,$$

$$i=1,2,\cdots,m; \ j=0,1,\cdots,n.$$

The homeomorphism T is cellular with respect to the above cellular decomposition and satisfies

$$T(C_i^{\pm}\times D_j)=(-1)^{i+j+1}(C_i^{\pm}\times D_j).$$

Let $\Phi : S^m \times CP(n) \to D(m,n)$ denote the projection, and write $(C_i, D_j) = \Phi(C_i^+ \times D_j)$. Then $\{(C_i, D_j) | i=0, 1, \cdots, m; j=0, 1, \cdots, n\}$ is a cellular decomposition of $D(m,n)$ whose boundary relations are given by

$$\partial(C_i, D_j) = (1+(-1)^{i+j})(C_{i-1}, D_j),$$
$$\partial(C_0, D_j) = 0,$$
$$i=1, 2, \cdots m; j=0, 1, \cdots, n,$$

and Φ is a cellular map. Let (c^i, d^j) denote the cochain dual to (C_i, D_j), then for the coboundary operation δ we have

$$\delta(c^i, d^j) = (1+(-1)^{i+j+1})(c^{i+1}, d^j).$$

From this we have

THEOREM. *The integral cohomology $H^*(D(m, m); Z)$ is a direct sum of the following groups:*

case m : even

free abelian group generated by (c^0, d^{2j}) and (c^m, d^{2j+1}), torsion group generated by (c^{2i}, d^{2j}) and (c^{2i-1}, d^{2j+1}) whose order are 2.

case m : odd

free abelian group generaated by (c^0, d^{2j}) and (c^m, d^{2j}), torsion group generated by (c^{2i}, d^{2j}) and (c^{2i-1}, d^{2j+1}) whose order are 2,

where $i=1, 2, \cdots, [m/2]$: $j=0, 1, \cdots, [n/2]$.

Dold's determination of the ring structure of $H^*(D(m,n); Z_2)$ can be described as follows:

THEOREM [2]. *The mod 2 cohomology ring $H^*(D(m,n); Z_2)$ is a truncated polynomial ring $Z_2[c, d]/(c^{m+1}, d^{n+1})$, where $c=(c^1, d^0)$ and $d=(c^0, d^1)$.*

We note that $H^2(D(m,n); Z) \simeq Z_2$ if $m \geqq 2$ with the generator reducing mod 2 to C^2.

4. The tangent bundle

The tangent bundle of a Dold manifold $D(m,n)$ was described by J. J. Ucci [5]. We summarize their results in this section, which are essential for our purpose.

Let $\ddot{\xi}$ be the canonical real line bundle over $RP(m)$, and let $\tilde{\eta}$ be the canonical complex line bundle over $CP(n)$. Let's represent a point of $D(m, n)$ by $[x, z]$ under the identification $(x, z)=(-x, \bar{\lambda}\bar{z})$ for $x \in S^m$, $z \in S^{2n+1} \subset C^{n+1}$ and all $\lambda \in S^1 \subset C$. Define a real 2-plane bundle η over $D(m, n)$ whose total space $E(\eta)$ is the set of all triples $[(x, z), \zeta]$ under the identification $((x, z), \zeta)=((-x, \bar{\lambda}\bar{z}), \overline{\lambda\zeta})$, where $\zeta \in C$, and x, z, and λ are as before, and define a projection map $p: E(\eta) \to D(m, n)$ by $p([(x, z), \zeta])=[x, z]$. For $m=0$, η is just the canonical complex line bundle $\tilde{\eta}$ over $D(0, n)=CP(n)$ considered as a real bundle, denoted $\mathrm{re}(\tilde{\eta})$; thus we obtain a bundle map (j, j_E)

$$\begin{array}{ccc} E(\mathrm{re}(\tilde{\eta})) & \xrightarrow{\ j_E\ } & E(\eta) \\ \downarrow p & & \downarrow p \\ D(0, n) & \xrightarrow{\ j\ } & D(m, n) \end{array}$$

implying $j^*\eta=\mathrm{re}(\tilde{\eta})$.

We define another line bundle ξ over $D(m, n)$ whose total space $E(\xi)$ is $S^m \times CP(n) \times R$ mod the identification $(x, z, t)=(-x, \bar{z}, -t)$. For $n=0$, ξ is just the canonical real line bundle $\tilde{\xi}$ over $D(m, 0)=RP(m)$, and so we obtain a bundle map (i, i_E)

$$\begin{array}{ccc} E(\tilde{\xi}) & \xrightarrow{\ i_E\ } & E(\xi) \\ \downarrow & & \downarrow \\ D(m, 0) & \xrightarrow{\ i\ } & D(m, n) \end{array}$$

implying that $i^*\xi=\tilde{\xi}$. In particular, by the naturality of characteristic classes, we have that the first Stiefel-Whitney class of $\xi: w_1(\xi)=c$, and that $w(\xi)=1+c$.

The map $S^m \times S^{2n+1} \times R^2 \to S^m \times S^{2n+1} \times C$ given by $(x, u; t_1, t_2) \to (x, u, t_1+it_2$ induces a bundle map (i, i_E')

$$\begin{array}{ccc} E(1 \oplus \tilde{\xi}) & \xrightarrow{\ i_E'\ } & E(\eta) \\ \downarrow & & \downarrow \\ D(m, 0) & \xrightarrow{\ i\ } & D(m, n) \end{array}$$

whence $i^*\eta=1 \oplus \tilde{\xi}$. So, $w_1(\eta)=c$.

The equivalences $i^*\eta=1+\tilde{\xi}$ and $j^*\eta=\mathrm{re}(\tilde{\eta})$ imply that $w_2(\eta)=d$ and so $w(\eta)=1+c+d$.

PROPOSITION. *There exist a 1-plane bundle ξ and a 2-plane bundle η over $D(m, n)$ such that*

i) $w(\xi)=1+c$, $w(\eta)=1+c+d$;

ii) $i^*\xi=\tilde{\xi}$, $j^*\eta=\mathrm{re}(\tilde{\eta})$, *and* $i^*\eta=1+\tilde{\xi}$

where $i: RP(m)\to D(m, n)$, $j: CP(n)\to D(m, n)$ *are the natural embeddings.*

Now we see a description of the tangent bundle $\tau=\tau(D(m, n))$ of $D(m, n)$.

THEOREM. $\tau(D(m,n))\oplus\xi\oplus 2=(m+1)\xi\oplus(n+1)\eta$.

PROOF. Write $\zeta=(m+1)\xi\oplus(n+1)\eta$ and $X=S^m\times S^{2n+1}\times R^{m+1}\times C^{n+1}$. Then $E(\zeta)$ is the set of all $(x,u,y,v)\in X$ mod the identifications $(x,u,y,v)\sim(x,\lambda u, y,\lambda v)\sim(-x,\bar{\lambda}\bar{u},-y,\bar{\lambda}\bar{v})\sim(-x,\bar{u},-y,\bar{v})$, where $\lambda\in C$ with $|\lambda|=1$.

Let $\langle\ ,\ \rangle$ and $(\ ,\ )$ denote the real and complex inner products of R^{m+1} and C^{n+1}, respectively. Then $E(\tau)$ is the subset of $E(\zeta)$ of all (x,u,y,v) satisfying $\langle x, y\rangle=0$ and $(u, v)=0$.

Since $E(\tau)\subset E(\zeta)$, we have $\tau\oplus\nu^3=\zeta$, where ν^3 (the orthogonal complement of τ in ζ) has total space $E(\nu^3)$ given by $\{(x,u,\alpha x,\beta u)\in X; \alpha\in R, \beta\in C\}$ mod the identification above. Now $E(2\oplus\xi)$ is given by $S^m\times CP(n)\times R^3$ mod the identifications $(x,u; t_1, t_2, t_3)\sim(x, \lambda u; t_1, t_2, t_3)\sim(-x, \bar{\lambda}\bar{u}; t_1, t_2, -t_3)\sim(-x, \bar{u}; t_1, t_2, -t_3)$ and so the map $S^m\times S^{2n+1}\times R^3\to E(\nu^3)$ given by $(x,u; t_1, t_2, t_3)\to(x, u; t_1x, (t_2+it_3)u)$ induces a bundle equivalence h_E

$$\begin{array}{ccc} E(\tau\oplus\xi) & \xrightarrow{\quad h_E \quad} & E(\nu^3) \\ \downarrow & & \downarrow \\ D(m, n) & = & D(m, n) \end{array}$$

Thus $\nu^3=\tau\oplus\xi$.

Now, we have the Stiefel-Whitney classes of $D(m, n)$.

THEOREM. $w(D(m, n))=(1+c)^m(1+c+d)^{n+1}$ (mod 2) *in* $H^*(D(m, n); Z_3)=Z_2[c, d]/(c^{m+1}, d^{n+1})$.

5. The Grothendieck ring of $D(m, n)$ and its parallelizability

Let F denote the field of real numbers R, or complex numbers C, and let X be a connected finite dimensional CW complex. The set $\mathrm{Vect}_F(X)$ of isomorphism classes of F-vector bundles over X admits a commutative semiring structure with the Whitney sum $\oplus$ and the tensor product $\otimes$. For the ring completion of this semiring, let's consider pairs $(\xi, \eta)\in \mathrm{Vect}_F(X)\times \mathrm{Vect}_F(X)$ and put the following equivalence relation on these pairs; (ξ, η) and (ξ', η') are equivalent provided there exists $\zeta\in \mathrm{Vect}_F(X)$ such that $\xi\oplus\eta'\oplus\zeta$ is isomorphic to $\xi'\oplus\eta\oplus\zeta$. Let $\xi-\eta$ denote the equivalence class of (ξ, η), and let $K_F(X)$ denote the set of equivalence classes. Then $K_F(X)$ is a commutative ring under

$$(\xi-\eta)+(\xi'-\eta')=(\xi\oplus\xi')-(\eta\oplus\eta'),$$

and

$$(\eta-\eta)\cdot(\xi'-\eta')=(\eta\otimes\xi')\oplus(\eta\otimes\eta')-(\eta\otimes\xi')\oplus(\xi\otimes\eta').$$

This ring $K_F(X)$ is called the *Grothendieck ring*, or *K_F-ring* of the space X. The rank function $rk: K_F(X)\to Z$ defined by $rk(\xi-\eta)=$dimension (ξ)-dimension (η) is clearly a ring homomorphism. Let $\tilde{K}_F(X)$ be the kernel of this homomorphism. Clearly, $K_F(X)=\tilde{K}_F(X)\oplus Z$, where a positive integer n represents the n-dimensional trivial F-vector bundle over X.

Two vector bundles ξ and η over a finite dimensional CW complex are called stably equivalent provided the Whitney sums $\xi\oplus n$ and $\eta\oplus m$ are isomorphic for some trivial bundles n and m. Stable equivalence is an equivalence relation, and the stable classes form a commutative ring under the Whitney sum and the tensor product. This is the $\tilde{K}$-ring of the space.

For a more precise description, let's consider a function α; $\mathrm{Vect}_F(X)\to \tilde{K}_F(X)$ defined by $\alpha(\xi)=\xi-\mathrm{rk}(\xi)$. For any ξ-$\eta\in\tilde{K}_F(\xi)$, where $\mathrm{rk}\xi=\mathrm{rk}\eta$, there exists a vector bundle η' such that $\eta\oplus\eta'$ is isomorphic to a trivial bundle, because the space X is a finite dimensional CW complex. Hence, $\xi-\eta=(\xi\oplus\eta')-(\eta\oplus\eta')=(\xi\oplus\eta')-\mathrm{rk}(\xi\oplus\eta')$. Consequently, α is surjective. Let $\xi-\mathrm{rk}(\xi)=\eta-\mathrm{rk}(\eta)$ in $\tilde{K}_F(X)$. Then, by the definition of $\tilde{K}_F(X)$, there exists a vector bundle ζ such that $\xi\oplus\mathrm{rk}(\eta)\oplus\zeta$ is isomorphic to $\eta\oplus\mathrm{rk}(\xi)\oplus\zeta$. Also, there exists a vector bundle ζ' such that $\zeta\oplus\zeta'$ is trivial, hence ξ and η are stably equivalent. Conversely, if ξ and η are stably equivalent, we see that $\alpha(\xi)=\alpha(\eta)$. We showed that $\tilde{K}_F(X)$ can be thought of as the ring of equivalence classes of the isomorphism classes of F-vector bundles over X under the stable equivalence relation.

Note that a space is stably parallelizable if and only if its tangent bundle is the zero element of its $\tilde{K}$-ring over the real field.

For any complex vector bundle ξ over X, let $\mathrm{re}(\xi)$ be the underlying real vector bundle of ξ, then re is a group homomorphism from $\tilde{K}_C(X)$ to $\tilde{K}_R(X)$.

(Note that it is not a ring homomorphism). Dually, for any real vector bundle η over X, let $c(\eta)$ be the complex vector bundle over X induced from η by taking the tensor product with C on each fibre of η, then c is a ring homomorphism from $\tilde{K}_R(X)$ to $\tilde{K}_C(X)$.

Clearly, there are the relations $\mathrm{re}(c(\eta))=2\eta$ and $c(\mathrm{re}(\xi))=\xi+\xi^*$, where ξ^* denote the conjugation of a complex vector bundle ξ.

To answer the stably parallelizability problem, recall that the 1-plane bundle ξ and the 2-plane η over $D(m, n)$ and that

$$\tau(D(m, n))\oplus\xi\oplus 2\simeq(m+1)\xi\oplus(n+1)\eta.$$

We write $x=\xi-1$, $z=\eta-2$ and $z=z-x$.
Ucci [5] showed

THEOREM. $\tilde{K}_R(D(m,n))$ *contains a summand isomorphic to* $Z_{2^{\phi(m)}}\oplus Z^{[n/2]}$ *generated by* x, y, $y^2, \cdots, y^h$ *with the relation* $2^{\phi(m)}x=0$, *where* $\phi(m)$ *is the numbers* s *such that* $0<s\leqq m$ *and* $s=0, 1, 2, 4 \pmod 8$.

Now let $D(m,n)$ be stably parallelizable, then $(D(m,n))-(m+2n)$ must be the zero element in $\tilde{K}_R(D(m,n))$, i.e.,

$$\begin{aligned} &m\xi+(n+1)\eta-\dim(m\xi+(n+1)\eta)\\ &=mx+(n+1)z\\ &=(m+n+1)x+(n+1)y \end{aligned}$$

must be zero in $\tilde{K}_R(D(m,n))$. Hence $n=0$ or 1. If $n=0$, then $m=1$, 3 or 7. If $n=1$, then $m+1=2^{\phi(m)}$, so that $m=0$, 2, 6. This gives the following main theorem.

THEOREM. *The Dold manifold* $D(m,n)$ *is stably parallelizable when and only when*

either i) $n=0$, *and* $m=1, 3$, *or* 7.

or ii) $n=1$ *and* $m=0$, 2, *or* 6.

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