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CURVATURE TENSORS OF 3-DIMENSIONAL ALMOST CONTACT METRIC MANIFOLDS

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1. 3-dimensional almost contact metric manifold

A $(2n+1)$-dimensional differentiable manifold M is called to have an almost contact metric structure if there is given a positive definite Riemannian metric g_{ji} and a triplet $(\varphi_k{}^j, \xi^j, \eta_k)$ of $(1,1)$-type tensor field $\varphi_k{}^j$, vector field ξ^j and 1-form η_k in M which satisfy the following equations:

$$(1.1) \qquad \varphi_j{}^i\varphi_i{}^k=-\delta_j^k+\eta_j\xi^k,\quad \varphi_j{}^i\xi^j=0,\quad \eta_i\varphi_j{}^i=0,\quad \eta_t\xi^t=1$$

and

$$(1.2) \qquad g_{ts}\varphi_j{}^t\varphi_i{}^s=g_{ji}-\eta_j\eta_i,\quad \eta_i=g_{it}\xi^t.$$

In this case, M is called a $(2n+1)$-dimensional almost contact metric menifold. By virtue of the last equation of (1.2), we shall write η^h instead of ξ^h.

We consider a $(0,4)$-type tensor E_{kjih} in a $(2n+1)$-dimensional almost contact metric manifold defined by

$$(1.3) \qquad E_{kjih}=(2n+1)(\gamma_{ki}\gamma_{jh}-\gamma_{ji}\gamma_{kh})-\varphi_{ki}\varphi_{jh}+\varphi_{ji}\varphi_{kh}-2\varphi_{kj}\varphi_{ih},$$

where we have put

$$(1.4) \qquad \gamma_{ki}=g_{ki}-\eta_k\eta_i.$$

By a direct computation, we obtain

$$(1.5) \qquad E_{kjih}E^{kjih}=16(2n+1)n(n^2-1).$$

Taking account of (1.3) and (1.5), we see that a (0.4)-type tensor E_{kjih} in a 3-dimensional almost contact metric manifold defined by

$$(1.6) \qquad E_{kjih}=3(\gamma_{ki}\gamma_{jh}-\gamma_{ji}\gamma_{kh})-\varphi_{ki}\varphi_{jh}+\varphi_{ji}\varphi_{kh}-2\varphi_{kj}\varphi_{ih}$$

is a zero tensor.

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Therefore we obtain in a 3-dimensional almost contact metric manifold M the following identity:

$$(1.7)\qquad 3(\gamma_{ki}\gamma_{jh}-\gamma_{ji}\gamma_{kh})=\varphi_{ki}\varphi_{jh}-\varphi_{ji}\varphi_{kh}+2\varphi_{kj}\varphi_{ih},$$

or equivalently

$$(1.8)\qquad \begin{aligned}&3(g_{ki}g_{jh}-g_{ji}g_{kh})+3\eta_k(g_{ji}\eta_h-g_{jh}\eta_i)-3\eta_j(g_{ki}\eta_h-g_{kh}\eta_i)\\&=\varphi_{ki}\varphi_{jh}-\varphi_{ji}\varphi_{kh}+2\varphi_{kj}\varphi_{ih}.\end{aligned}$$

Thus we have the following

THEOREM 1.1. *In a 3-dimensional almost contact metric manifold, the identity* (1.7) *or equivalently* (1.8) *is satisfied.*

On the other hand, it is well known fact that the conformal curvature tensor of Weyl vanishes identically in a 3-dimensional Riemannian manifold, that is, the following equation is satisfied:

$$(1.9)\qquad \begin{aligned}&K_{kji}{}^h+K_{ki}\delta_j^h-K_{ji}\delta_k^h+g_{ki}K_j{}^h-g_{ji}K_k{}^h\\&-\frac{K}{2}(g_{ki}\delta_j^h-g_{ji}\delta_k^h)=0,\end{aligned}$$

where $K_{kji}{}^h$, K_{ji} and K are the curvature tensor, the Ricci tensor and the scalar curvature of the manifold respectively.

We call the section determined by a unit vector v^h orthogonal to η^h and the vector $\varphi_t{}^h v^t$ a *φ-holomorphic section* in a 3-dimensional almost contact metric manifold.

2. Curvature tensor of 3-dimensional cosymplectic manifold

If an almost contact metric structure of M introduced in the last section satisfies

$$N_{ji}{}^h+(\partial_j\eta_i-\partial_i\eta_j)\eta^h=0,$$

where $\partial_j=\partial/\partial x^j$, $\{U,x^h\}$ being the coordinate neighborhoods and $N_{ji}{}^h$ is the Nijenhuis tensor formed with $\varphi_j{}^h$, then M is called a normal almost contact metric maifold.

A normal almost contact metric manifold M is said to be cosymplectic if the 2-form $\varphi_{ji}=\varphi_j{}^t g_{ti}$ and the 1-form η_i are both closed. It is well known (Blair, [1]) that the cosymplectic structure of M is characterized by

$$(2.1)\qquad \nabla_k\varphi_j{}^h=0,\quad \nabla_k\eta_j=0,$$

where ∇_k is the operator of covariant differentiation with respect to g_{ji}. The following equations are also satisfied in cosymplectic manifold M:

$$K_{kji}{}^{t}\eta_t=0, \quad K_{jt}\eta^t=0 \tag{2.2}$$

and

$$\eta^t\nabla_t K_{ji}=0, \quad \eta^t\nabla_t K=0. \tag{2.3}$$

In this section, we study on the curvature tensor of a 3-dimensional cosymplectic manifold M.

Transvecting (1.9) with $\eta_h\eta^k$ and taking account of (2.2), we obtain

$$K_{ji}=\frac{K}{2}\gamma_{ji}. \tag{2.4}$$

Substituting (2.4) into (1.9), we see that the curvature tensor of M has the form

$$K_{kjih}=\frac{K}{2}(\gamma_{kh}\gamma_{ji}-\gamma_{jh}\gamma_{ki}), \tag{2.5}$$

or equivalently

$$K_{kjih}=\frac{K}{6}(\varphi_{kh}\varphi_{ji}-\varphi_{ki}\varphi_{jh}-2\varphi_{kj}\varphi_{ih}) \tag{2.6}$$

by virtue of (1.7).

Thus we have the following

THEOREM 2.1. *The curvature tensor of the 3-dimensional cosymplectic manifold has the form* (2.5) *or equivalently* (2.6).

On the other hand, on previous paper (Eum, [2]), we have defined the cosymplectic Bochner curvature tensor in a 3-dimensional cosymplectic manifold by

$$B_{kjih}=K_{kjih}+\gamma_{kh}L_{ji}-\gamma_{jh}L_{ki}+L_{kh}\gamma_{ji}-L_{jh}\gamma_{ki}+\varphi_{kh}M_{ji}-\varphi_{jh}M_{ki}+M_{kh}\varphi_{ji}-M_{jh}\varphi_{ki}-2(M_{kj}\varphi_{ih}+\varphi_{kj}M_{ih}), \tag{2.7}$$

where

$$L_{ji}=-\frac{1}{6}\left(K_{ji}-\frac{K}{8}\gamma_{ji}\right), \qquad L=L_{ji}g^{ji}=-\frac{K}{8}, \tag{2.8}$$

$$M_{ji}=-L_{jt}\varphi_i{}^t. \tag{2.9}$$

Substituting (2.4) into (2.8), we obtain

$$L_{ji}=-\frac{K}{16}\gamma_{ji} \tag{2.10}$$

and from which

$$(2.11) \qquad M_{ji} = -\frac{K}{16}\varphi_{ji}.$$

Substituting (2.5), (2.10) and (2.11) into (2.7), we obtain

$$(2.12) \qquad B_{kjih} = 0.$$

Thus we have the following

THEOREM 2.2. *In the 3-dimensional cosymplectic manifold, the cosymplectic Bochner curvature tensor vanishes identically.*

Taking account of the equation (2.5) and calculating the sectional curvature k determind by the φ-holomorphic section, we easily see that $k = \frac{K}{2}$.

Taking account of this fact, (1.7), (2.5) and (2.6), we find

$$(2.13) \qquad K_{kjih} = \frac{k}{4}(\gamma_{kh}\gamma_{ji} - \gamma_{jh}\gamma_{ki} + \varphi_{kh}\varphi_{ji} - \varphi_{jh}\varphi_{ki} - 2\varphi_{kj}\varphi_{ih}).$$

Thus we have the following

THEOREM 2.3. *In the 3-dimensional cosymplectic manifold, the φ-holomorphic sectional curvature is independent of φ-holomorphic section at a point and is equal to $\frac{K}{2}$, K being the scalar curvature.*

3. Curvature tensor of 3-dimensional Sasakian manifold

If a normal almost contact metric structure of M satisfies

$$(3.1) \qquad \varphi_{ji} = \frac{1}{2}(\partial_j\eta_i - \partial_i\eta_j),$$

then M is called a Sasakian manifold.

In a $(2n+1)$-dimensional Sasakian manifold M, we have (Yano, [6])

$$(3.2) \qquad \nabla_k\eta_j = \varphi_{kj}, \quad \nabla_k\varphi_j{}^h = -\eta^h g_{kj} + \eta_j\delta_k^h.$$

The following equations are also satisfied in M (Yano, [6]):

$$(3.3) \qquad K_{kjt}{}^h\eta^t = \delta_k^h\eta_j - \delta_j^h\eta_k, \quad \eta^t K_{tji}{}^h = \eta^h g_{ji} - \eta_i\delta_j^h,$$

$$(3.4) \qquad K_{jt}\eta^t = 2n\eta_j$$

and

(3.5) $$\eta^t \nabla_t K_{ji}=0, \quad \eta^t \nabla_t K=0.$$

In this section we study on the curvature tensor of the 3-dimensional Sasakian manifold M.

Transvecting (1.9) with $\eta_h \eta^k$ and taking account of (3.3) and (3.4), we obtain

(3.6) $$K_{ji}=bg_{ji}+(2-b)\eta_j\eta_i,$$

where we have put

(3.7) $$b=\frac{K}{2}-1.$$

Substituting (3.6) and (3.7) into (1.9), we see that the curvature tensor of M has the form

(3.8) $$\begin{aligned} K_{kji}{}^h=&(1-b)(g_{ki}\delta_j^h-g_{ji}\delta_k^h) \\ &+(b-2)\{(\eta_k\delta_j^h-\eta_j\delta_k^h)\eta_i+(g_{ki}\eta_j-g_{ji}\eta_k)\eta^h\}. \end{aligned}$$

Substituting the identity (1.8) into (3.8), we obtain

(3.9) $$\begin{aligned} K_{kjih}=&(g_{ki}g_{jh}-g_{ji}g_{kh})+2\eta_k(g_{ji}\eta_h-g_{jh}\eta_i) \\ &-2\eta_j(g_{ki}\eta_h-g_{kh}\eta_i)-\frac{b}{3}(\varphi_{ki}\varphi_{jh}-\varphi_{ji}\varphi_{kh}+2\varphi_{kj}\varphi_{ih}). \end{aligned}$$

Substituting (1.7) into (3.9), we obtain

(3.10) $$\begin{aligned} K_{kjih}=&(1-b)(\gamma_{ki}\gamma_{jh}-\gamma_{ji}\gamma_{kh}) \\ &+\eta_k(g_{ji}\eta_h-g_{jh}\eta_i)-\eta_j(g_{ki}\eta_h-g_{kh}\eta_i). \end{aligned}$$

Thus we have the following

THEOREM 3.1. *The curvature tensor of the 3-dimensional Sasakian manifold has the form* (3.8) *or* (3.9) *or equivalently* (3.10).

On the other hand, the contact Bochner curvature tensor in the 3-dimensional Sasakian manifold is defined by (Yano, [6])

(3.11) $$\begin{aligned} B_{kji}{}^h=&K_{kji}{}^h+\gamma_k{}^hL_{ji}-\gamma_j{}^hL_{ki}+L_k{}^h\gamma_{ji}-L_j{}^h\gamma_{ki} \\ &+\varphi_k{}^hM_{ji}-\varphi_j{}^hM_{ki}+M_k{}^h\varphi_{ji}-M_j{}^h\varphi_{ki} \\ &-2(M_{kj}\varphi_i{}^h+\varphi_{kj}M_i{}^h)+(\varphi_k{}^h\varphi_{ji}-\varphi_j{}^h\varphi_{ki}-2\varphi_{kj}\varphi_i{}^h), \end{aligned}$$

where

(3.12) $$\begin{aligned} &L_{ji}=-\frac{1}{6}\{K_{ji}+(L+3)g_{ji}-(L-1)\eta_j\eta_i\}, \quad L_j{}^h=L_{jt}g^{th}, \\ &M_{ji}=-L_{jt}\varphi_i{}^t, \quad M_k{}^h=M_{kt}g^{th}, \end{aligned}$$

and

(3.13) $$L=-\frac{K+10}{8}=-\left(\frac{b}{4}+\frac{3}{2}\right).$$

Substituting (3.6) and (3.7) into (3.12) and (3.13), we obtain

(3.14) $$L_{ji}=b_1 g_{ji}+b_2\eta_j\eta_i, \quad M_{ji}=b_1\varphi_{ji},$$

where we have put

(3.15) $$b_1=-\frac{1}{2}\left(\frac{1}{4}b+\frac{1}{2}\right), \quad b_2=\frac{1}{2}\left(\frac{1}{4}b-\frac{3}{2}\right).$$

Substituting (1.7), (3.10), (3.14) and (3.15) into (3.11), we easily obtain

(3.16) $$B_{kji}{}^h=0.$$

Thus we have the following

THEOREM 3.2. *In the 3-dimensional Sasakian manifold M, the contact Bochner curvature tensor vanishes identically.*

Taking account of the definition of φ-holomorphic sectional curvature and the equation (3.10), we easily see the following

THEOREM 3.3. *In the 3-dimensional Sasakian manifold, the φ-holomorphic sectional curvature is iindependent of φ-holomorphic section at a point and is equal to $\frac{K}{2}-2$, K being the scalar curvature.*

Taking account of above fact, (1.8) and (3.9), we obtain

$$K_{kjih}=\frac{k+3}{4}(g_{kh}g_{ji}-g_{jh}g_{ki})-\frac{k-1}{4}\{\eta_k(g_{ji}\eta_h-g_{jh}\eta_i) -\eta_j(g_{ki}\eta_h-g_{kh}\eta_i)+\varphi_{ki}\varphi_{jh}-\varphi_{ji}\varphi_{kh}+2\varphi_{kj}\varphi_{ih}\},$$

k being the φ-holomorphic sectional curvature.

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