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FIXED POINT THEOREMS OF GENERALIZED NONEXPANSIVE MAPS

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1. Introduction

Let (C, d) be a metric space. A map $T: C \rightarrow C$ is said to be *generalized nonexpansive*. if, for any $x, y \in C$,

(1) $$d(Tx, Ty) \leq a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty) + a_4 d(x, Ty) + a_5 d(y, Tx),$$

where $a_i \geq 0$ and $\sum_{i=1}^{5} a_i \leq 1$.

In [10], Goebel, Kirk and Shimi proved that if T is a continuous generalized nonexpansive selfmap of a nonempty closed convex bounded subset C of a uniformly convex Banach space, then T has a fixed point. Bogin [4] generalized this result for a weakly compact convex subset C having normal structure in a Banach space without assuming the continuity of T. In this paper, we study various types of maps which are particular to (1) by classifying a_i's, and obtain several new fixed point theorems. Especially, in section 2, we classify a_i's, and study various cases of (1) without assuming normal structure. In section 3, we generalize the result of [4] for the case that C has asymptotic normal structure, and obtain new fixed point theorems for some variations of (1), and common fixed point theorems for a commuting family of generalized nonexpansive maps.

2. Generalized nonexpansive maps

By interchanging x and y, (1) is equivalent to the condition

(2) $$d(Tx, Ty) \leq ad(x, y) + b\{d(x, Tx) + d(y, Ty)\} + c\{d(x, Ty) + d(y, Tx)\},$$

for all $x, y \in C$, where $a, b, c \geq 0$ and $a + 2b + 2c \leq 1$, by putting $a = a_1$,

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$b=(a_2+a_3)/2$, and $c=(a_4+a_5)/2$.

If $a+2b+2c<1$ and C is complete, a number of authors showed that T has a unique fixed point, and any iteration $\{T^n x\}$ converges to the fixed point of T for each $x\in C$. Therefore, we may assume that $a+2b+2c=1$. Then the following cases can be occurred:

Case I. $a=1$, $b=c=0$.

Case II. $a=c=0$, $b=\frac{1}{2}$.

Case III. $b>0$, $c>0$.

Case IV. $a>0$, $b>0$, $c=0$.

Case V. $b=0$, $c>0$ (this case contains the case $a=b=0$).

Case I. $a=1$, $b=c=0$. In this case, the map T is said to be *nonexpansive*. In 1965,

Browder [6] showed that any nonexpansive selfmap of a closed convex bounded subset C of a Hilbert space has a fixed point by using monotone operator theory. Also, in the same year,

Kirk [16] obtained the same result for a closed convex bounded subset C of a reflexive Banach space provided that C has normal structure.

The concept of normal structure was introduced by Brodskii and Mil'man [5]. A nonempty closed convex bounded subset C of a Banach space is said to have *normal structure* if, for any closed convex subset C_0 of C which has more than one point, there exists a point $x\in C_0$ satisfying

$$\sup\{\|x-y\|;\ y\in C_0\}<\delta(C_0),$$

where $\delta(C_0)$ denotes the diameter of the set C_0.

Moreover, if C is a weakly compact convex subset of a Banach space and C has asymptotic normal structure, then T has a fixed point by [2]. But, Alspach [1] gave an example of a weakly compact convex subset of $L_1[0,1]$ that fails to have the fixed point property for nonexpansive maps.

If T is affine, that is, $T(\lambda x+(1-\lambda)y)=\lambda Tx+(1-\lambda)Ty$, $0\leq\lambda\leq 1$, then T is weakly continuous. Therefore, if C is weakly compact and convex, and if T is an affine nonexpansive selfmap of C, then T has a fixed point by the Tychonoff fixed point theorem. Otherwise, if we impose a condition similar to that T is affine, then we have the following

THEOREM 2. 1. *Let C be a nonempty weakly compact convex subset of a Banach space, and $T: C \to C$ be nonexpansive. Suppose that there is a strictly increasing continuous function $\gamma: [0, \infty) \to [0, \infty)$ with $\gamma(0)=0$ such that, for all $x, y \in C$ and $0 \leq \lambda \leq 1$,*

(3) $\|T(\lambda x+(1-\lambda)y)-\lambda x-(1-\lambda)y\| \leq \gamma(\|x-Tx\|+\|y-Ty\|)$.

Then T has a fixed point.

Proof. Choose a decreasing sequence $\{\varepsilon_n\}$ of positive reals such that $\varepsilon_n \to 0$ as $n \to \infty$, and $\gamma(2\varepsilon_{n+1}) \leq \varepsilon_n$ for $n \geq 1$. By the Banach contraction principle, we can choose $x_n \in C$ such that $\|Tx_n - x_n\| \leq \varepsilon_n$. Since C is weakly compact, we may assume that x_n converges weakly to a point x in C.

Call a sequence $\{y_n\}$ a c-subsequence (see [12]) of $\{x_n\}$ provided that there is a sequence of integers $1=p_1 \leq q_1 < p_2 \leq q_2 < \ldots$ and coefficients $\alpha_i \geq 0$ such that

$$\sum_{i=p_n}^{q_n} \alpha_i = 1, \quad y_n = \sum_{i=p_n}^{q_n} \alpha_i x_i$$

Since every closed convex subset of a Banach space is weakly closed, we may choose a c-subsequence $\{y_n\}$ of $\{x_n\}$ such that y_n converges strongly to x. Then, by (3), and by using induction, we get

$$\|Ty_n - y_n\| = \|T\left(\sum_{i=p_n}^{q_n} \alpha_i x_i\right) - \sum_{i=p_n}^{q_n} \alpha_i x_i\| \leq \gamma(2\varepsilon_{p_n}) \leq \varepsilon_{p_n-1}.$$

By setting $n \to \infty$, we have $\lim \|Ty_n - y_n\| = 0$, so that x is a fixed point of T.

REMARK 2. 1. Under the same hypothesis of Theorem 2. 1, we know that every weak cluster point of $\{S^n x\}$ is a fixed point of T for any $x \in C$, where $S_\lambda = \lambda I + (1-\lambda)T$, $0 < \lambda < 1$, since, by Ishikawa [14], S_λ is asymptotically regular. Note that every affine map satisfies (3). Also note that every nonexpansive selfmap of a closed convex bounded subset of a uniformly convex Banach space satisfies (3) by Bruck [7]. Actually, he showed that if T is a nonexpansive selfmap of a closed convex bounded subset C of a uniformly convex Banach space, then there is a strictly increasing continuous function $\alpha: [0, \infty) \to [0, \infty)$ with $\alpha(0)=0$ such that, for all $x, y \in C$, and $0 \leq \lambda \leq 1$,

(3)′ $\alpha(\|\lambda Tx + (1-\lambda)Ty - T(\lambda x + (1-\lambda)y)\| \leq \|x-y\| - \|Tx - Ty\|$.

Since C is bounded, we may assume that $\alpha(t) \to \infty$ as $t \to \infty$, so that α^{-1} exists. Therefore by putting $r(t)=\alpha^{-1}(t)+t$, T satisfies (3). Moreover, note that we can easily construct a nonexpansive map which satisfies (3), but not (3).

By putting $\phi(x)=\|x-Tx\|$, $x\in C$, in Theorem 2.1, the existence of a fixed point of T is equivalent to the fact that ϕ attains its minimum. It is well-known that if ϕ is convex, then ϕ attains its minimum. Note that the condition (3) is a weakened form of the convexity of ϕ.

Case II. $a=c=0$, $b=\frac{1}{2}$. In this case, T is called a Kannan-type map. In 1973, Kannan [15] showed that every nonempty closed convex bounded subset of a reflexive Banach space having normal structure has the fixed point property for Kannan type maps. Note that every nonexpansive map is continuous, while a Kannan-type map need not be continuous. The existence of fixed points of Kannan-type maps related to cose-to-normal structure. A closed convex bounded subset C of a Banach space is said to have *close-to-normal* structure if, for each closed convex subset C_0 of C having more than one point, there exists $x\in C_0$ such that $\|x-y\|<\delta(C_0)$ for any $y\in C_0$. In [24], Wong showed that any nonempty weakly compact convex subset C of a Banach space has the fixed point property for Kannan-type maps if and only if C has close-to-normal structure. Furthermore, he posed a question whether every closed convex bounded subset of a reflexive Banach space has close-to-normal structure. But

Tan [22] showed that the anwer is negative by giving an example of a reflexive Banach space which has asymptotic normal structure, but does not have close-to-normal structure.

Now we have the following

THEOREM 2.2. *Any reflexive Banach space X admits an equivalent norm $\|\ \|_1$ such that any selfmap T of a nonempty closed convex bounded subset C of X satisfying, for $x, y\in C$,*

$$\|Tx-Ty\|_1\leq\frac{1}{2}\{\|x-Tx\|_1+\|y-Ty\|_1\}$$

has a unique fixed point.

Proof. By Troyanski [23], X admits an equivalent norm $\|\ \|_1$ so that $(X, \| \|_1)$ is locally uniformly convex. Therefore with the new norm $\| \|_1$, C has close-to-normal structure, and by [24] T has a unique fixed point.

REMARK 2.2. Note that Dulst [9] showed that every separable Banach space X admits an equivalent norm such that every nonexpansive selfmap (with the new norm) of a weakly compact convex subset of X has a fixed point. Since this new norm satisfies the Opial condition [20], the result of [9] can be applied to Kannan-type maps. Also note that every separable Banach space has an equivalent norm which is strictly convex, so that every closed convex bounded subset has close-to-normal structure.

Case III. $b>0$, $c>0$. In this case, Bogin [4] showed that if C is a complete metric space, then any iteration $\{T^n x\}$ converges to the unique fixed point of T.

Case IV. $a>0$, $b>0$, $c=0$. In this case, Gregus [11] showed that if C is a closed convex subset of a Banach space, then T has a unique fixed point. Actually, he proved that any iteration $\{U^n x\}$ converges to the unique fixed point of T, where $Ux=(T^2x+T^3x)/2$.

Case V. $b=0$, $c>0$. In this case, we have the following lemma.

LEMMA 2.1. *Let (C,d) be a bounded metric space, and let $T:C\to C$ be a map satisfying*

(4) $$d(Tx, Ty)\le ad(x,y)+cd(x,Ty)+cd(y,Tx)$$

for all $x,y\in C$, where $a\ge 0$, $c>0$ and $a+2c=1$. Then T is asymptotically regular, i.e., for any $x\in C$,

$$\lim d(T^{n+1}x, T^n x)=0.$$

Proof. For $x_0\in C$, let $x_n=T^n x_0$. Then, for $n\ge 1$, by (4), we get

$$\begin{aligned} d(x_{n+1}, x_n) &= d(Tx_n, Tx_{n-1}) \\ &\le ad(x_n, x_{n-1})+cd(x_n, Tx_n)+cd(x_{n-1}, Tx_{n-1}), \end{aligned}$$

so that we have

$$d(x_{n+1}, x_n)\le \frac{a+c}{1-c} d(x_n, x_{n-1})=d(x_n, x_{n-1}).$$

Therefore, the sequence $\{d(x_{n+1}, x_n)\}$ is nonincreasng, so that $\lim d(x_{n+1}, x_n)=r$ exists. We must show that $r=0$. Suppose $r>0$. Then there exists a positive integer s such that the diameter of $C=d_1<(s+$

$1)r/2$. Since $c>0$, there exists $\varepsilon>0$ such that $\{1-(s+1)c^s\}(r+\varepsilon)+(s+1)rc^s/2<r$ (this is possible for $0<\varepsilon\leq(s+1)rc^s/2)$. Then there exists a positive integer N such that $n\geq N$ implies $r\leq d(x_{n+1}, x_n)<r+\varepsilon$.

Now we claim that, for $n\geq 0$ and $k\geq 1$

$$(5)\quad d(x_{n+k+1}, x_{n+k})\leq\{1-(k+1)c^k\}d(x_{n+1}, x_n)+c^k d(x_{n+k+1}, x_n).$$

To prove (5), let $k=1$. Then. we get, by (4).

$$d(x_{n+2}, x_{n+1})\leq ad(x_{n+1}, x_n)+cd(x_{n+2}, x_n),$$

which asserts (5), since $a=1-2c$. In order to use induction for k, assume that (5) is true for $k\geq 1$. Then we have

$$\begin{aligned}d(x_{n+k+2}, x_{n+k+1})&\leq\{1-(k+1)c^k\}d(x_{n+2}, x_{n+1})+c^k d(x_{n+k+2}, x_{n+1})\\&\leq\{1-(k+1)c^k\}d(x_{n+1}, x_n)\\&\quad+c^k\{ad(x_{n+k+1}, x_n)+cd(x_{n+k+2}, x_n)+cd(x_{n+k+1}, x_{n+1})\}\\&\leq\{1-(k+1)c^k\}d(x_{n+1}, x_n)\\&\quad+c^k\{a(k+1)+ck\}d(x_{n+1}, x_n)+c^{k+1}d(x_{n+k+2}, x_n)\\&\leq\{1-(k+2)c^{k+1}\}d(x_{n+1}, x_n)+c^{k+1}d(x_{n+k+2}, x_n),\end{aligned}$$

by using $d(x_{n+k+1}, x_n)\leq(k+1)d(x_{n+1}, x_n)$ and $d(x_{n+k+1}, x_{n+1})\leq kd(x_{n+1}, x_n)$, which proves(5).

Then, for $n\geq N$ and $k=s$, by (5), we have

$$\begin{aligned}d(x_{n+s+1}, x_{n+s})&\leq\{1-(s+1)c^s\}(r+\varepsilon)+c^k d_1\\&\leq\{1-(s+1)c^s\}(r+\varepsilon)+\frac{c^k(s+1)r}{2}\\&<r,\end{aligned}$$

which leads a contradiction. Therefore, we have $r=0$.

In Lemma 2.1, if $a=0$, then we have stronger conclusion that T is uniformly asymptotically regular, that is, $d(T^{n+1}x, T^n x)\to 0$ uniformly as $n\to\infty$ for all $x\in C$.

LEMMA 2.2. *Let (C, d) be a bounded metric space, and let $T: C\to C$ be a map satisfying*

$$(6)\qquad d(Tx, Ty)\leq\frac{1}{2}\{d(x, Ty)+d(y, Tx)\}$$

for all $x, y\in C$. Then T is uniformly asymptotically regular. Moreover we have

$$(7)\qquad d(T^{n+1}x, T^n x)\leq\frac{1}{2}\cdot\frac{3}{4}\cdots\frac{2n-1}{2n}\delta(C),$$

where $\delta(C)$ is the diameter of the set C.

Proof. Since

$$\frac{1}{2}\cdot\frac{3}{4}\cdots\frac{2n-1}{2n}=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\sin^{2n}x\,dx$$

converges to 0 as $n\to\infty$, we need only to prove (7).

First we claim that, for $n\geqslant 2$,

(8) $$d(T^n x,\ Tx)\le\frac{1}{2}d(T^n x, x)+\frac{1}{2^2}d(T^{n-1}x, x)+\dots+\frac{1}{2^{n-1}}d(T^2 x, x).$$

To prove (8), for $n=2$, by (6), we have

$$d(T^2 x,\ Tx)\le\frac{1}{2}d(Tx,\ Tx)+\frac{1}{2}d(T^2 x,\ x)=\frac{1}{2}d(T^2 x,\ x).$$

Assume that (8) is true for $n-1\ge 2$. Then by the assumption and (6), we obtain

$$\begin{aligned}d(T^n x,\ Tx)&\le\frac{1}{2}d(T^n x,\ x)+\frac{1}{2}d(T^{n-1}x,\ Tx)\\&\le\frac{1}{2}d(T^n x,\ x)\\&\quad+\frac{1}{2}\left\{\frac{1}{2}d(T^{n-1}x,\ x)+\cdots+\frac{1}{2^{n-2}}d(T^2 x,\ x)\right\}\\&=\frac{1}{2}d(T^n x,\ x)+\frac{1}{2^2}d(T^{n-1}x, x)+\cdots+\frac{1}{2^{n-1}}d(T^2 x, x),\end{aligned}$$

which asserts (8) by induction.

Next we shall prove that, for $n\ge 1$,

(9) $$d(T^{n+1}x,\ T^n x)\le\frac{a_{n+1,1}}{2^n}d(T^{n+1}x,\ x)+\frac{a_{n+1,2}}{2^{n+1}}d(T^n x,\ x)+\dots+\frac{a_{n+1,n}}{2^{2n-1}}d(T^2 x,\ x),$$

where $a_{n+1,k}$'s $(1\le k\le n)$ are inductively given by the rule;

$$a_{2,1}=1$$

and, for $n\ge 2$,

$$\begin{aligned}&a_{n+1,1}=a_{n,1},\\&a_{n+1,2}=a_{n,1}+a_{n,2},\\&\dots\dots\dots\dots\\&a_{n+1,n-1}=a_{n,1}+a_{n,2}+\dots+a_{n,n-1},\\&a_{n+1,n}=a_{n,1}+a_{n,2}+\dots+a_{n,n-1}.\end{aligned}$$

To prove (9), assume that (9) is true for $n-1\geq 1$. Then, by (8), we have

$$\begin{aligned} d(T^{n+1}x,\ T^n x) &\leq \frac{a_{n,1}}{2^{n-1}} d(T^{n+1}x,\ Tx) + \dots + \frac{a_{n,n-1}}{2^{2n-3}} d(T^2x, Tx) \\ &\leq \frac{a_{n,1}}{2^{n-1}} \left\{ \frac{1}{2} d(T^{n+1}x,\ x) + \dots + \frac{1}{2^n} d(T^2x, x) \right\} \\ &\quad + \dots + \frac{a_{n,n-1}}{2^{2n-3}} \left\{ \frac{1}{2} d(T^3x, x) + \frac{1}{2^2} d(T^2x, x) \right\} \\ &= \frac{a_{n+1,1}}{2^n} d(T^{n+1}x,\ x) + \dots + \frac{a_{n+1,n}}{2^{2n-1}} d(T^2x, x), \end{aligned}$$

which proves (9) by induction.

Now, by elementary calculus, it is easy to see that

$$\frac{a_{n+1,1}}{2^n} + \frac{a_{n+1,2}}{2^{n+1}} + \dots + \frac{a_{n+1,n}}{2^{2n-1}} = \frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-1}{2n}.$$

Therefore, by (9), we complete the proof of (7).

Using the above lemmas, we have the following

THEOREM 2.3. *Let (C, d) be a compact metric space, and let $T: C \to C$ be a map satisfying (4). Then T has a fixed point, and any iteration $\{T^n x\}$ converges to a fixed point of T for each $x \in C$.*

Proof. For each $x \in C$, there exists a subsequence $\{T^{n_i}x\}$ of $\{T^n x\}$ which converges to some p in C, by the compactness of C. Then, by (4), we get

$$\begin{aligned} d(Tp,\ T^{n_i}x) &\leq ad(p,\ T^{n_i-1}x) + cd(p,\ T^{n_i}x) + cd(Tp,\ T^{n_i-1}x) \\ &\leq ad(p,\ T^{n_i}x) + ad(T^{n_i}x,\ T^{n_i-1}x) + cd(p,\ T^{n_i}x) \\ &\quad + cd(Tp,\ T^{n_i}x) + cd(T^{n_i}x,\ T^{n_i-1}x), \end{aligned}$$

so that we obtain

$$\begin{aligned} d(Tp,\ T^{n_i}x) &\leq \frac{a+c}{1-c} \{d(p,\ T^{n_i}x) + d(T^{n_i}x,\ T^{n_i-1}x)\} \\ &= d(p,\ T^{n_i}x) + d(T^{n_i}x,\ T^{n_i-1}x). \end{aligned}$$

Therefore we have

$$\begin{aligned} d(Tp,\ p) &= \lim_{i\to\infty} d(Tp,\ T^{n_i}x) \\ &\leq \lim_{i\to\infty} \{d(p,\ T^{n_i}x) + d(T^{n_i}x,\ T^{n_i-1}x)\}. \end{aligned}$$

Since the left hand side of the above inequality tends to 0 as $i \to$

∞ by Lemma 2.1, we have $Tp=p$.

Moreover, by (4), we have

$$d(T^{n+1}x, p) \leq ad(T^n x, p) + cd(T^n x, p) + cd(T^{n+1}x, p),$$

so that we obtain

$$d(T^{n+1}x, p) \leq \frac{a+c}{1-c} d(T^n x, p) = d(T^n x, p).$$

Therefore, the sequence $\{d(T^n x, p)\}$ is nonincreasing, and the whole sequence $\{T^n x\}$ converges to p.

COROLLARY. *Let (C, d) be a compact metric space, and let $T: C \to C$ be a map satisfying (6). Then T has a fixed point, and any iteration $\{T^n x\}$ converges to a fixed point of T, for each $x \in C$.*

REMARK 2.3. If (C, d) is not bounded, then Lemma 2.1 and 2.2. are not valid. For example, consider a map $T: R \to R$ such that $Tx = x + a$ $(a \neq 0)$ with the usual metric on R. Then Tsatisfies (4) and (6), but $d(T^{n+1}x, T^n x) = |a|$ for all $x \in R$ and all $n \geq 1$.

Also note if C is a weakly compact convex subset of a Banach space and C has normal structure, and if T is a selfmap of C satisfying (4), then T has a fixed point by [4]. In section 2, we shall extend this result for the case that C has asymptotic normal structure. But the following example shows that the conditions on C are indispensable.

EXAMPLE. Let $C[0, 1]$ be a Banach space of all continuous real valued functions on $[0, 1]$ with the uniform norm. Let $C = \{f \in C[0, 1];\ 0 \leq f \leq 1,\ f(0) = 0,\ f(1) = 1\}$. Then C is a closed convex bounded subset of $C[0, 1]$. Define a map $T: C \to C$ by $Tf(x) = xf(x)$. Then T is nonexpansive. Also we can prove that T satisfies (6), and so that T satisfies (4) for all $a, c \geq 0$ with $a + 2c = 1$. But T has no fixed point.

3. Asymptotic Normal Structure and Fixed Points

Recall that a closed convex bounded subset C of a Banach space has *asymptotic normal structure* (see [2]) if, for each closed convex subset C_0 of C consisting of more than one point and each sequence $\{x_n\}$ in C_0 satisfying $x_{n+1} - x_n \to 0$ as $n \to \infty$, there is a point $x \in C_0$ such that $\liminf \|x_n - x\| < \delta(C_0)$. Note that if C has normal structure, then C has asymptotic normal structure. But the converse is not true.

Now, we state our main result in this section.

THEOREM 3.1. *Let C be a nonempty weakly compact convex subset of a Banach space. Suppose that C has asymptotic normal structure. Let $T: C \to C$ be a map satisfying*

$$\|Tx-Ty\| \leq a\|x-y\| + b\{\|x-Tx\|+\|y-Ty\|\} + c\{\|x-Ty\|+\|y-Tx\|\}$$

for all $x, y \in C$, with $a, c \geq 0$, $0 \leq b < \frac{1}{2}$ and $a+2b+2c=1$. Then T has a fixed point.

Proof. If $b=c=0$, then the theorem is true by Baillon and Schöneberg [2]. If $b>0$, $c>0$, then the theorem is valid by Bogin [4]. If $b>0$, $c=0$, then also the theorem remains true by Gregus [11]. Therefore we need only to prove the theorem for the case $b=0$ and $c>0$, so that we may assume that T satisfies (4).

By the standard Zorn's Lemma argument using weak compactness of C, there exists a nonempty weakly compact convex subset C_0 of C which is minimal in the sense that it contains no proper closed convex subset which is invariant under T.

Now we claim that C_0 is a singleton, whose element is a fixed point of T. Suppose that C_0 has more than one point.

Let x_0 be any fixed element in C_0, and let $x_n = T^n x_0$. Then by Lemma 2.1 $x_{n+1} - x_n \to 0$ as $n \to \infty$. Next we claim that, for each $x \in C_0$,

$$\lim \|x_n - x\| = \delta(C_0). \tag{10}$$

Therefore, we have a contradiction to the asymptotic normal structure of C. To prove (10), let $y \in C_0$, and let $s = \limsup \|x_n - y\|$. Let $D = \{x \in C_0;\ \limsup \|x_n - x\| \leq s\}$, which is nonempty closed and convex. Then, by (4), we have

$$\begin{aligned} \|Tx - x_n\| &= \|Tx - Tx_{n-1}\| \\ &\leq a\|x - x_{n-1}\| + c\|x - Tx_{n-1}\| + c\|x_{n-1} - Tx\| \\ &\leq (a+c)\|x - x_n\| + c\|x_n - Tx\| + (a+c)\|x_n - x_{n-1}\|. \end{aligned}$$

so that we obtain

$$\begin{aligned} \|Tx - x_n\| &\leq \frac{a+c}{1-c}\|x - x_n\| + \frac{a+c}{1-c}\|x_n - x_{n-1}\| \\ &= \|x - x_n\| + \|x_n - x_{n-1}\|, \end{aligned}$$

which shows that D is invariant under T. By the minimality of C_0, $D=C_0$. Choose a subsequence $\{x_{n_i}\}$ so that $\lim\|x_{n_i}-y\|=s'$ exists. Suppose that there exists z in C_0 and a subsequence $\{x_{n_j}\}$ of $\{x_{n_i}\}$ such that $\lim\|x_{n_j}-z\|=t$. Let $E=\{x\in C_0;\ \lim\sup\|x_{n_j}-x\|\leq\min\{t, s'\}\}$. Repeating the above argument, we find $E=C_0$. Therefore $y, z\in C_0=E$, and so $t=s'$. Thus, for each $x\in C_0$, $\lim\|x_{n_j}-x\|$ exists and equals s'.

We complete the proof by showing that $s'=r=\delta(C_0)$. From this it follows that $\|x_{n_i}-y\|\to r$ whenever $\{\|x_{n_i}-y\|\}$ converges. Therefore by the boundedness of $\{x_n\}$, $\|x_n-y\|\to r$ for the entire sequence.

For this purpose, consider $F=\{u\in C_0;\ \sup\{\|u-x\|;\ x\in C_0\}\leq s'\}$. Then F is nonempty because we can choose a weakly convergent subsequence, again denoted by $\{x_{n_i}\}$ with the limit z. Since $\|x_{n_i}-x\|\to s'$ for each $x\in C_0$, it follows that $\|x-z\|\leq s'$, so that $z\in F$. Now if $s'<r$, then F is a proper closed convex subset of C_0. However, this contradicts the minimality of C_0 because F is invariant under T. To see the latter, let w be in F, and let $\sup\{\|Tw-x\|;\ x\in C_0\}=s_1$. We must prove that $s_1\leq s'$. Suppose $s_1>s'$. Choose ε with $0<\varepsilon<(a+c)(s_1-s')/2$. Then there exists $u\in C_0$ such that $s_1<\|Tw-u\|+\varepsilon$. By the minimality of C_0, we can see that the closed convexhull of TC_0 is actually C_0. Therefore we can choose $v=\sum_{i=i}^{k}\lambda_i Tv_i$ with $v_i\in C_0$, $\lambda_i>0$, $\sum_{i=1}^{k}\lambda_i=1$, and $\|u-v\|\leq\varepsilon$. Then we have

$$\begin{aligned}
s_1&<\|Tw-u\|+\varepsilon\\
&\leq\|Tw-v\|+\|v-u\|+\varepsilon\\
&\leq\sum_{i=1}^{k}\lambda_i\|Tw-Tv_i\|+2\varepsilon\\
&\leq\sum_{i=1}^{k}\lambda_i\{a\|w-v_i\|+c\|w-Tv_i\|+c\|v_i-Tw\|\}+2\varepsilon\\
&\leq\sum_{i=1}^{k}\lambda_i(as'+cs'+cs_1)+2\varepsilon\\
&=(a+c)s'+cs_1+2\varepsilon<s_1,
\end{aligned}$$

which is a contradiction. Therefore, we have $s_1\leq s'$. This completes the proof.

REMARK 3.1. Theorem 3.1 is a generalization of [4] and [10] except for the case $b=1/2$. However, the conclusion of Theorem 3.1 does not hold for the case $b=1/2$ by [22].

In Theorem 3.1, instead of asymptotic normal structure of C, let us consider close-to-normal structure (cf. [13]).

THEOREM 3.2. *Let C be a nonempty weakly compact convex subset of a Banach space. Suppose that C has close-to-normal structure, and that* $T: C \to C$ *is a generalized nonexpansive map with* $b>0$ *in* (2). *Then T has a unique fixed point.*

Proof. From [4], [11] and [24], the proof is clear.

Next we consider a family of generalized nonexpansive maps and their common fixed points. For this end, we need the following lemma.

LEMMA 3.1. ([19]). *Let C be a closed convex subset of a strictly convex Banach space, and let* $T: C \to C$ *be a generalized nonexpansive map with* $a>0$ *in* (2). *Then the set* $F(T)$ *of all fixed points of T is closed and convex.*

THEOREM 3.3. *Let C be a nonempty weakly compact convex subset of a strictly convex Banach space. Suppose that C has asymptotic normal structure, and that* $\mathfrak{F}$ *is an arbitrary commuting family of selfmaps of C such that each member of* $\mathfrak{F}$ *satisfies* (2) *with* $a>0$. *Then* $\mathfrak{F}$ *has a common fixed point, i.e., there is a point* $p \in C$ *such that* $Tp=p$ *for every* $T \in \mathfrak{F}$.

Proof. By Theorem 3.1, each member T of $\mathfrak{F}$ has a nonempty fixed point set $F(T)$. Moreover, by Lemma 3.1, each $F(T)$ is a closed convex subset of C, so that it is weakly compact. Let $\mathcal{A}=\{F(T);\ T \in \mathfrak{F}\}$. Now we claim that $\mathcal{A}$ satisfies the finite intersection condition, so that $\bigcap \{F(T);T \in \mathfrak{F}\}$ is nonempty, and it is the set of common fixed points of $\mathfrak{F}$.

Suppose $T_1, T_2, \ldots, T_n \in \mathfrak{F}$. Since $T_1T_2=T_2T_1$, $F(T_1)$ is invariant under T_2. Therefore T_2 has a fixed point in $F(T_1)$, so that $F(T_1) \cap F(T_2)$ is nonempty, closed and convex. Since T_3 commutes T_1 and T_2, $F(T_1) \cap F(T_2)$ is invariant under T_3, so that T_3 has a fixed point in $F(T_1) \cap F(T_2)$. Therefore $F(T_1) \cap F(T_2) \cap F(T_3)$ is nonempty, closed and convex. By the same argument and by induction, we have $\bigcap_{i=1}^{n} F(T_i) \neq \phi$, which shows that $\mathcal{A}$ satisfies the finite intersection condition.

By the same line as in the proof of Theorem 3.3, we have the following

THEOREM 3.4. *Let C be a nonempty weakly compact convex subset of a Banach space. Suppose that C has close-to-normal structure, and that $\mathfrak{F}$ is a commuting family of selfmaps of C such that each member of $\mathfrak{F}$ satisfies* (2) *with $b>0$. Then $\mathfrak{F}$ has a unique common fixed point. In particular, if one of members in $\mathfrak{F}$ satisfies* (2) *with $b>0$, then $\mathfrak{F}$ has a unique common fixed point.*

REMARK 3.2. For a family $\mathfrak{F}$ of nonexpansive maps, a number of authors ([3], [8], [17], [18] and [21]) investigated common fixed points of $\mathfrak{F}$. Note that every nonexpansive map is continuous, but a generalized nonexpansive map need not be continuous in general. Therefore, in a strictly convex Banach space, Theorem 3.3 is a generalization of [3], [17], [18] and [21].

Finally we shall prove two fixed point theorems which are concerned with variations of (2). At first, we have the following

THEOREM 3.5. *Let C be a nonempty weakly compact convex subset of a Banach space, and let T be a selfmap of C satisfying*

$$\|Tx-Ty\| \leq \max\{\|x-y\|,\ \frac{1}{2}\|x-Ty\|+\frac{1}{2}\|y-Tx\|\},$$

for all $x, y \in C$. If C has normal structure, then T has a fixed point.

Proof. By Zorn's lemma, we have a nonempty minimal closed convex subset C_0 of C in the sense that it contains no proper closed convex subset which is invariant under T. Suppose that C_0 has more than one point. For an arbitrary $x_0 \in C_0$, put $x_n = T^n x_0$. Then, by Bogin [4], there exists $r>0$ such that $C_1 = \{x \in C_0;\ \lim\sup \|x_n - x\| \leq r\}$ is a nonempty proper closed convex subset of C_0. Let $x \in C_1$ and $\lim\sup \|x_n - Tx\| = r_0$. Since

$$\|x_n - Tx\| = \|Tx_{n-1} - Tx\| \leq \max\{\|x_{n-1}-x\|,\ \frac{1}{2}\|Tx_{n-1}-x\|+\frac{1}{2}\|x_{n-1}-Tx\|\},$$

we have

$$\begin{aligned} r_0 &= \lim\sup \|Tx - x_n\| \\ &\leq \lim\sup \{\max\{\|x_{n-1}-x\|,\ \frac{1}{2}\|x_n - x\|+\frac{1}{2}\|x_{n-1}-Tx\|\}\} \\ &\leq \max\left\{r,\ \frac{1}{2}r+\frac{1}{2}r_0\right\}. \end{aligned}$$

Therefore, we have $r_0 \leq r$, and so that $Tx \in C_1$, which shows that C_1 is invariant under T. This is a contradiction. Therefore C_0 is a singleton.

Let (C, d) be a complete metric space. We define the *Kuratowski measure of noncompactness* α as a nonnegative real valued function on the set of all bounded subsets of C such that $\alpha(D) = \inf\{r > 0; D$ is covered by finitely many sets with diameter less than $r\}$. It is well-known that $\alpha(D) = 0$ if and only if the closure $\bar{D}$ of D is compact.

A map $T: C \to C$ is said to be *condensing* if, for each bounded subset D of C, TD is bounded and

$$\alpha(TD) < \alpha(D) \text{ for all } \alpha(D) \neq 0.$$

Note that we do not assume that T is continuous.

THEOREM 3.6. *Let (C, d) be a bounded complete metric space, and let $T: C \to C$ be a generalized nonexpansive and condensing (not necessarily continuous) map with $c > 0$ in (2). Then T has a fixed point, and any iteration $\{T^n x\}$ converges to a fixed point of T for each $x \in C$.*

Proof. By Lemma 2.1 and [4], T is asymptotically regular. Let x_0 be an arbitrary point in C, and let $x_n = T^n x_0$. Then we claim that $\alpha(\{x_n\}_{n=0}^{\infty}) = 0$. Suppose $\alpha(\{x_n\}_{n=0}^{\infty}) > 0$. Since T is condensing, we have

$$\alpha(\{x_n\}_{n=0}^{\infty}) = \alpha(T\{x_n\}_{n=1}^{\infty}) < \alpha(\{x_n\}_{n=0}^{\infty}).$$

But $\alpha(\{x_n\}_{n=0}^{\infty}) = \max\{\alpha(\{x_0\}), \alpha(\{x_n\}_{n=1}^{\infty})\}$ gives that the above inequality is a contradiction. Therefore, $\{x_n\}$ is relatively compact, so that we can choose a convergent subsequence $\{x_{n_i}\}$ with the limit p. Then, by (2), we have

$$d(Tp, x_{n_i}) \leq ad(p, x_{n_i-1}) + b\{d(p, Tp) + d(x_{n_i-1}, x_{n_i})\} + c\{d(p, x_{n_i}) + d(x_{n_i-1}, Tp)\},$$

so that we obtain

$$d(Tp, x_{n_i}) \leq \frac{a+c}{1-c} d(p, x_{n_i}) + \frac{a+b+c}{1-c} d(x_{n_i-1}, x_{n_i}) + bd(p, Tp).$$

By letting $i \to \infty$, we have $d(Tp, p) \leq bd(p, Tp)$, so that $Tp = p$. By the same way in the proof of Theorem 2.3 we can prove that $\{T^n x_0\}$ converges to p.

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