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CHARACTERIZATIONS OF ORDER IDEALS AND PERFECT SUBSPACES IN THE ORDERED NORMED SPACE OF $n\times n$ HERMITION MATRICES

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1. Preliminaries

In an ordered locally convex space, subspaces can be classified by various order properties. These order properties along with their interrelations are studied in detail in another paper by the author [1].

In this paper, we specifically consider the ordered normed space consisting of all the $n\times n$ Hermitian matrices, and give characterizations of order ideals and perfect subspaces. By these characterizations we get a concrete insight of what an order ideal or a perfect subspace should be and how they can be constructed.

We will denote E to be the real normed space of $n\times n$ Hermitian matrices, where the norm of a matrix is taken to be the maximum of the absolute values of its eigenvalues. If K is the set of matrices A in E whose eigenvalues are all nonnegative, i. e, $\overline{X}^T AX\geq 0$ for all n-vector X, then K is a generating cone in E. we take K to be the positive cone of E.

An elenent $P\in V\cap K$, where V is the closed unit ball of E, is called an extreme point of $V\cap K$ if whenever $P=\lambda Q+(1-\lambda)\ R$ for some Q, $R\in V\cap K$, $0<\lambda<1$, we must have $P=Q=R$. It is easy to verify that if P is an extreme point, then $0\leq Q\leq P$ implies $Q=\lambda P$ for some $0\leq\lambda\leq 1$.

We denote e_k to be the unit vector whose all entries are zero except the kth one of value 1. We will denote E_{kl} for $e_k e_l^T+e_l e_k^T$ and $\hat{E}_{kl}$ for $ie_k e_l^T-ie_l e_k^T$.

A subspace J is called a perfect subspace or a nearly directed subspace [2] if for any $0\leq a\in J$, $\varepsilon>0$, there exists $b\in J$ such that 0, $a\leq b+\varepsilon$.

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where ε in this case is ε times the identity matrix. We will be using the fact that a subspace J is a perfect subspace of E if and only if J° is an order ideal in E' [3].

2. A Characterization of extreme points

LEMMA 2.1. *Let U be a unitary matrix, i.e. $\bar{U}^T U=1$. Then $A\geq 0$ implies $\bar{U}^T A U\geq 0$.*

Proof. Let X be an arbitrary n-vector. Then $\bar{X}^T(\bar{U}^T A U)X=(\overline{UX})^T A(UX)\geq 0$ since $A\geq 0$.

LEMMA 2.2. *Let U be a unitary matrix and let $P\geq 0$. Then P is an extreme point of $V\cap K$ if and only if $\bar{U}^T P U$ is an extreme point of $V\cap K$.*

Proof. Assume P is an extreme point of $V\cap K$. By Lemma (2.1), $\bar{U}^T P U\in V\cap K$. Suppose $\bar{U}^T P U=\lambda Q+(1-\lambda)R$ for some $Q, R\in V\cap K$ and for $0<\lambda<1$. Then $P=\lambda UQ\bar{U}^T+(1-\lambda)UR\bar{U}^T$ with $UQ\bar{U}^T$, $UR\bar{U}^T\in V\cap K$. Hence $P=UQ\bar{U}^T=UR\bar{U}^T$ which in turn implies $\bar{U}^T P U=Q=R$. A similar proof for the if part is omitted.

LEMMA 2.3. *Let $D\geq 0$ be a diagonal matrix. Then D is an extreme point of $V\cap K$ if and anly if $D=e_k e_k^T$ for some $1\leq k\leq n$.*

Proof. First, we consider the if part. In view of Lemma (2.2), we may assume $D=e_1 e_1^T$ without loss of generality. If $D=\lambda P+(1-\lambda)Q$ for some $P, Q\in V\cap K$ and $0<\lambda<1$, then $e_l^T D e_l=\lambda e_l^T P e_l+(1-\lambda)e_l^T Q e_l$. Hence $0=\lambda p_{ll}+(1-\lambda)q_{ll}\ \forall\ l\neq 1$. Since $p_{ll}\geq 0$, $q_{ll}\geq 0$, we must have $p_{ll}=q_{ll}=0\ \forall l\neq 1$. Now, if $k\neq l$ and if k and l are different from 1, then from

$$(e_k+e_l)^T P(e_k+e_l)=p_{kl}+p_{lk}=p_{kl}+\bar{p}_{kl}\geq 0$$
$$(e_k-e_l)^T P(e_k-e_l)=-p_{kl}-p_{lk}=-(p_{kl}+\bar{p}_{kl})\geq 0,$$

we must have Re $(p_{kl})=0$. Similarly, by applying ie_k+ie_l we obtain Im $(p_{kl})=0$. Thus, we have

$$p_{lk}=p_{kl}=0 \quad l, k\gneq l \quad l\neq k.$$

On the other hand, from $0\leq\lambda P\leq D$, we have

$$0\leq\lambda(e_1+\alpha e_l)^T P(e_1+\alpha e_l)\leq(e_1+\alpha e_l)^T e_1 e_1^T(e_1+\alpha e_l)\ \forall l\neq 1$$

Hence, $\lambda(p_{11}+\alpha p_{1l}+\alpha p_{l1})\leq 1\ \forall\alpha\in R$. Thus $\mathrm{Re}(p_{1l})=0$. Similarly by applying $e_1+i\alpha e_l$, we obtain Im $(p_{1l})=0$. Therefore, we conclude

$P=p_{11}e_1e_1^T$.

Next, we consider the only if part of the proof. Let $D=(d_k)$ with $d_l\neq 0$ for some l. Then clearly we have $0\leq d_le_le_l^T\leq D$.

Since D is an extreme point, $d_le_le_l^T=\lambda D$ for some $0\leq\lambda\leq 1$. Now, $D=\alpha e_le_l^T$ for some $\alpha\geq 0$ implies $d_k=0\ \forall\ k\neq l$ and hence $d_l=\lambda=1$.

THEOREM 2.4. *Let $P\in V\cap K$. Then P is an extreme point of $V\cap K$ if and only if $P=\bar{U}^Te_ke_k^TU$ for some unitary matrix U and for some $1\leq k\leq n$.*

Proof. If part follows from Lemma (2.3) and Lemma (2.2). For the only if part, recall that every Hermitian matrix is diagonalizable and hence $P=\bar{U}^TDU$ for some unitary matrix U and diagonal matrix D. Now by Lemma (2.2), D is an extreme point. Hence the proof follows from Lemma (2.3).

COROLLARY 2.5. *Let $P\in V\cap K$. Then P is an extreme point if and only if $P=X\bar{X}^T$ for some n-vector X with norm 1.*

Proof. Only if part follows directly from Theorem (2.4) by setting $X=\bar{U}^Te_k$. For the if part, note that $X\bar{X}^T$ is a matrix whose characteristic equation is of the form $(\lambda-1)\lambda^{n-1}$, and eigenvectors are X and $(n-1)$ mutually orthogonal vectors each of which is orthogonal to X. If we form a matrix U from these eigenvectors, $X=Ue_1$. Hence $P=Ue_1(\bar{U}e_1)^T=Ue_1e_1^T\bar{U}^T$. Now, the proof follows from Theorem (2.4).

COROLLARY 2.6. *Let $P\in V\cap K$. Then P is an extreme point of $V\cap K$ if and only if the characteristic equation of P has a single root of 1 and the other $(n-1)$ roots are all zeros.*

3. A characterization of order ideals

Lemma 3.1. *Let J be a positively generated order ideal of E. Then there exists $Q\in J\cap K$ such that J is generated by Q, i.e, J is the smallest order ideal containing Q.*

Proof. Let $\mathcal{J}=\{J_p\,|\,J_p$ is the order ideal generated by $P\in J\cap K\}$. Then $\mathcal{J}$ is partially ordered by set inclusion. Since E is a finite dimensional vector space, evey totally ordered subset of $\mathcal{J}$ is finite and has a supremum. Thus, by Zorn's Lemma, there is a maximal element J_o. Let P_o be the generator of J_o, then clearly $J_o \subseteq J$.

Suppose $J_o \neq J$, then there exists $P \in J \cap K$ with $P \notin J_o$. But then the order ideal generated by $P_o + P$ contains J_o, which is a contradiction to the maximality of J_o.

LEMMA 3.2. *Let I_m be the diagonal matrix whose first m diagonal elements are* 1 *and others are* 0, *and let J be the order ideal generated by I_m. Then $J = E_m$.*

Proof. Since E_m is a positively generated order ideal containing I_m, it is clear that $J \subseteq E_m$. To show the converse, let $A \in E_m$. We can write $A = \bar{U}^T D U$ for a diagonal $D \in E_m$, and a unitary matrix U of the form

$$U = \begin{pmatrix} U_m & 0 \\ 0 & I_{n-m} \end{pmatrix}$$

where U_m is unitary in E_m and I_{n-m} is the identity matrix of order $n-m$.

Let D^+ be the diagonal matrix whose entries are the same as the corresponding entries in D if they are positive and zero otherwise. Then

$$0, \ D \leq D^+ \leq \|A\| I_m$$

And similarly we have

$$0, \ -D \leq (-D)^+ = D^- \leq \|A\| I_m.$$

Therefore, $-\|A\| I_m \leq D \leq \|A\| I_m$ which implies

$$-\|A\| I_m \leq \bar{U}^T D U \leq \|A\| I_m, \text{ and hence } A \in J.$$

LEMMA 3.3. *Let $P \in K$ and let J be the order ideal generated by P. Then there exists a unitary matrix U and $m \leq n$ such that $J = \bar{U}^T E_m U$.*

Proof. Let $P = \bar{U}^T D U$ where D is diagonal and U is a unitary matrix. Without loss of generality, we may assume that the diagonal elements d_k of D satisfies $d_k = 0 \ \forall \ k > m$ and $d_k \neq 0 \ \forall k \leq m$.

By (3.2), the order ideal generated by D is E_m since $\lambda I_m \leq D$ for $\lambda = \min\{d_k | K = 1, 2 \ldots m\} \gneq 0$. Therefore, the order ideal generated by $P = \bar{U}^T D U$ is $\bar{U}^T E_m U$

THEOREM 3.4. *Let J be a positively generated subspace. Then J is an order ideal if and only if there exists a unitary matrix U and $m \leq n$ such that $J = \bar{U}^T E_m U$.*

Proof. If part of the Theorem is clear. For the only if part, use Lemma (3.1) to pick a generator P of J. Then the theorem follows from Lemma (3.3).

LEMMA 3.5. *Let* $H_{mo}=\{A\in E\,|\,a_{ij}=0\ \forall\ i,j\leq m,\ a_{ii}=0\ \forall\ i\geq m+1\}$, *and let* L *be a subspace of* H_{mo}. *If* $J=E_m+L$, *then* $E_m=J\cap K-J\cap K$, *where* E_m *is the subspace of all* $m\times m$ *Hermitian matrices considered as an imbedded subspace of* E.

Proof. If is clear that $E_m\subseteq J\cap K-J\cap K$. Hence, it is left to show $J\cap K\subseteq E_m$. Take an arbitrary element A of $J\cap K$, then $A=B+C$ for some $B\in E_m$ and $C\in L$. If $k\geq m+1$,

$$\begin{aligned}(e_k+\bar{\lambda}e_l)^T A(e_k+\lambda e_l) &= a_{kk}+\lambda\bar{\lambda}a_{ll}+\lambda a_{kl}+\bar{\lambda}a_{lk}\\ &= b_{kk}+\lambda\bar{\lambda}b_{ll}+\lambda b_{kl}+\bar{\lambda}\bar{b}_{kl}+c_{kk}+\lambda\bar{\lambda}c_{ll}+\lambda c_{kl}+\bar{\lambda}\bar{c}_{kl}\\ &= \lambda\bar{\lambda}b_{ll}+\lambda c_{kl}+\bar{\lambda}c_{kl}\geq 0\ \forall\ \text{complex number } \lambda.\end{aligned}$$

Now, by choosing suitable value for λ, we get $c_{kl}=0\ \forall\ k\geq m+1$. Hence $C=0$, which implies $A=B\in E_m$.

COROLLARY 3.6. *If* H_{mo} *and* L *are as in Lemma* (3.5), *and if* $J=E_m+L$, *then* J *is an order ideal.*

Proof. J is an order ideal if and only if $J\cap K-J\cap K$ is an order ideal. But the latter is the same subspace as E_m by Lemma (3.5), and E_m is a positively generated order ideal in E. Therefore J is an order ideal.

LEMMA 3.7. *If* J *is an order ideal such that* $J\cap K-J\cap K=E_m$ *for some* $m<n$, *then* $J=E_m+L$ *where* L *is a subspace of* H_{mo} *defined in Lemma* (3.5).

Proof. First, we prove the case when $m=n-1$. Enough to prove that for an arbitrary element $A\in J$, $a_{nn}=0$. Assume that $a_{nn}\neq 0$. Without loss of generality, We may assume that $a_{nn}>0$, $|a_{in}|\leq 1$, $i=1,2\ldots\ldots n$. Let

$$B=\begin{pmatrix}\lambda & & 0 & & a_{1n}\\ & \lambda & & & a_{2n}\\ 0 & & \ddots & & \\ & & & \lambda & \\ & & & & \lambda\\ a_{n1} & a_{n2} & \cdots & & a_{nn}\end{pmatrix}$$

where λ is chosen such that $\lambda > \frac{16n}{a_{nn}}$. Then it is clear that $B \in J$ and $B \notin E_m$.

We now prove that $B \geq 0$, i. e, $B \in K$ which would contradict the hypothesis $J \cap K - J \cap K = E_m$.

Let X be an arbitrary n-vector such that $\sum_{k=1}^{n} |x_k|^2 = 1$.

Then

$$\bar{X}^T B X = \lambda \sum_{i=1}^{n-1} |x_i|^2 + \left(\sum_{i=1}^{n-1} \bar{x}_i a_{in}\right) x_n + \left(\sum_{i=1}^{n} a_{in} x_i\right) \bar{x}_n$$

(1) when $|x_n| \geq \sqrt{1 - \left(\frac{a_{nn}}{4n}\right)^2}$

Since $\sum_{i=1}^{n} |x_i|^2 = 1$, we have

$$|x_i| \leq \sqrt{1 - |x_n|^2} \leq \sqrt{1 - 1 + \left(\frac{a_{nn}}{4n}\right)^2} = \frac{a_{nn}}{4n}, \quad i = 1, 2, \ldots\ldots n-1,$$

Thus,

$$\sum_{i=1}^{n-1} |x_i| \leq \frac{a_{nn}}{4} \quad \text{and}$$

$$2\left(\sum_{i=1}^{n-1} |x_i|\right) |x_n| \leq \frac{a_{nn}}{2} |x_n| \leq a_{nn} |x_n|^2 \quad \textit{from}$$

$$|x_n| \geq \sqrt{1 - \left(\frac{1}{4n}\right)^2} \geq \frac{1}{2}.$$

Therefore,

$$\bar{X}^T B X \geq a_{nn} |x_n|^2 + \lambda (1 - |x_n|^2) - 2\left(\sum_{i=1}^{n-1} |x_i|\right) |x_n| \geq 0$$

(2) When $|x_n| \leq \sqrt{1 - \left(\frac{a_{nn}}{4n}\right)^2}$

From

$$|x_n|^2 \leq 1 - \left(\frac{a_{nn}}{4n}\right)^2, \quad \text{we have} \quad \left(\frac{a_{nn}}{4n}\right)^2 \leq 1 - |x_n|^2$$

and hence

$$1 \leq \left(\frac{4n}{a_{nn}}\right)^2 (1 - |x_n|^2).$$

By multiplying both sices by $1 - |x_n|^2$,

$$1 - |x_n|^2 \leq \left(\frac{4n}{a_{nn}}\right)^2 (1 - |x_n|^2)^2$$

and hence,

$$\sqrt{1-|x_n|^2} \leq \frac{4n}{a_{nn}}(1-|x_n|^2).$$

Now, Since

$$\sum_{k=1}^{l} |x_k| \leq 2\sqrt{\sum_{k=1}^{l} |x_k|^2} \quad \text{for all} \quad l=1, 2\ldots\ldots,$$

We have

$$\begin{aligned} 2\left(\sum_{i=1}^{n-1} |x_i|\right)|x_n| &\leq 4\left(\sqrt{\sum_{i-1}^{n-1} |x_i|^2}\right)|x_n| \\ &= 4\sqrt{1-|x_n|^2} \cdot |x_n| \\ &\leq \frac{16n}{a_{nn}}(1-|x_n|^2)|x_n| \leq \lambda(1-|x_n|^2) \end{aligned}$$

Therefore,

$$\begin{aligned} \bar{X}^T BX &= \lambda \sum_{i=1}^{n-1} |x_i|^2 + \sum_{i=1}^{n-1} (\bar{x}_i a_{in}) x_n + \left(\sum_{i=1}^{n-1} a_{in} x_i\right)\bar{x}_n + a_{nn}|x_n|^2 \\ &\geq \lambda(1-|x_n|^2) - 2\left(\sum_{i=1}^{n-1} |x_i|\right)|x_n| + a_{nn}|x_n|^2 \\ &\geq \lambda(1-|x_n|^2) - 2\left(\sum_{i=1}^{n-1} |x_i|\right)|x_n| \geq 0. \end{aligned}$$

Next, we consider the case when $m<n-1$. Note that $J_{m+1}=J\cap E_{m+1}$ is an order ideal in E_n since both J and E_{m+1} are order ideals in E. Hence for any element $A \in J$, $a_{m+1,m+1}=0$. Now, by mathematical induction $a_{k,k}=0 \ \forall \ k \geq m+1$.

THEOREM 3.8. *Let J be an order ideal in E. Then there exists a unitary matrix U such that $J=\bar{U}^T(E_m+L)U$ for some $m \leq n$,* where L *is a subspace of H_{mo}.*

Proof. Let $J_1=J\cap K-J\cap K$. Then J_1 is a positively generated order ideal. Hence by Theorem (3.4), there exists a unitary matrix U such that

$$J_1=\bar{U}^T E_m U \quad \text{for some} \quad m \leq n$$

Now, $E_m=UJ_1\bar{U}^T$. If $J_2=UJ\bar{U}^T$ then J_2 is an order ideal since so is J. Furthermore, $J_2\cap K-J_2\cap K=E_m$ and hence by Lemma (3.7), J_2 is of the form E_m+L where L is a subspace of H_{mo}.

COROLLARY 3.9. *Let J be a subspace of E. Then J is an order ideal if and only if there exists a unitary matrix U such that $J=\bar{U}^T(E_m+L)$*

U for some $m\leq n$ and a subspace L of H_{mo}.

4. A characterization of perfect subspaces

PROPOSITION 4.1. *Let f be a real linear functional on E. Then there exists $G\in E$ such that $f(A)=\frac{1}{2}Tr(GA+AG)$ $\forall A\in E$*

Proof. Let

$$E_{kl}=e_k e_l^T+e_l e_k^T$$
$$\hat{E}_{k,l}=ie_k e_l^T-ie_l e_k^T$$
$$f_{k,l}=\frac{1}{2}f(E_{k,l}),\quad \hat{f}_{k,l}=\frac{1}{2}f(\hat{E}_{k,l})$$

and define G to be an $n\times n$ matrix whose (k,l)-element is $f_{k,l}+i\hat{f}_{k,l}$. Then it is easy to check that G is Hermitian. Now, if we let $A_{k,l}=a_{k,l}+i\hat{a}_{k,l}$ with $a_{k,l}$ and $\hat{a}_{k,l}$ real then

$$\begin{aligned}f(A)&=f\left(\sum_{k=1}^{n}\frac{1}{2}a_{kk}E_{kk}+\sum_{k>l}a_{k,l}E_{kl}+\sum_{k>l}\hat{a}_{k,l}\hat{E}_{k,l}\right)\\&=\sum_{k=1}^{n}a_{kk}f_{kk}+2\sum_{k>l}a_{k,l}f_{kl}+2\sum_{k>l}\hat{a}_{kl}\hat{f}_{kl}\\&=\sum_{k,l}a_{kl}\cdot f_{kl}+\sum_{k,l}\hat{a}_{kl}\cdot\hat{f}_{kl},\quad\text{where } \hat{a}_{kk}=0\end{aligned}$$

and

$$\begin{aligned}(GA)_{kk}&=\sum_{l=1}^{n}(f_{kl}+i\hat{f}_{kl})(a_{lk}+i\hat{a}_{lk})\\&=\sum_{l=1}^{n}(f_{kl}a_{lk}-\hat{f}_{kl}\hat{a}_{lk})+i\sum_{l=1}^{n}(f_{kl}\hat{a}_{lk}+\hat{f}_{kl}a_{lk})\\(AG)_{kk}&=\sum_{l=1}^{n}(a_{kl}+i\hat{a}_{kl})(f_{lk}+i\hat{f}_{lk})\\&=\sum_{l=1}^{n}(a_{kl}f_{lk}-\hat{a}_{kl}\hat{f}_{lk})+i\sum_{l=1}^{n}(a_{kl}\hat{f}_{lk}+\hat{a}_{kl}f_{lk})\\&=\sum_{l=1}^{n}(f_{kl}a_{lk}-\hat{f}_{kl}\hat{a}_{lk})-i\sum_{l=1}^{n}(\hat{f}_{kl}a_{lk}+f_{kl}\hat{a}_{lk})\end{aligned}$$

Hence $(GA)_{kk}+(AG)_{kk}=2\sum_{l=1}^{n}(f_{kl}a_{lk}-\hat{f}_{kl}\hat{a}_{lk})$

$$\begin{aligned}\frac{1}{2}Tr(GA+AG)&=\sum_{k=1}^{n}\sum_{l=1}^{n}(f_{kl}a_{lk}-\hat{f}_{kl}\hat{a}_{lk})\\&=\sum_{k,l}(\hat{f}_{kl}a_{kl}+\hat{f}_{kl}\hat{a}_{kl})=f(A)\end{aligned}$$

LEMMA 4.2. (a) $Tr(AE_{\alpha\beta}+E_{\alpha\beta}A)=4a_{\alpha\beta}$
(b) $Tr(A\hat{E}_{\alpha\beta}+\hat{E}_{\alpha\beta}A)=4\hat{a}_{\alpha\beta}$, *where* $A_{\alpha\beta}=a_{\alpha\beta}+i\hat{a}_{\alpha\beta}$

Proof. A routine proof of this is omitted.

LEMMA 4.3. *Let* $H_m=\{A\in E \mid A_{ij}=0\ \forall\ i,j\leq m\}$. *Then* $H_m=E_m^o$

Proof. To prove $H_m\subseteq E_m^o$, let $F\in H_m$ be an arbitrary element, then $F_{ij}=0\ \forall i,j\leq m$. Now, let $A\in E_m$ then $A_{ji}=A_{ij}=0\ \forall i\geq m+1$. Hence $F_{kj}\ A_{jk}=0\ \forall_{j,k}$ since if $k\geq m+1$ then $A_{jk}=0\ \forall j=1,2\ldots n$ and if $k\leq m$ then $F_{kj}=0$ for $j\leq m$ and $A_{jk}=0$ for $j\geq m+1$. Thus,

$$(FA)_{kk}=\sum_{j=1}^{n} F_{kj}A_{jk}=0$$

and similarly,

$$(AF)_{kk}=\sum_{j=1}^{n} A_{kj}F_{jk}=0$$

Therefore,

$$Tr(AF+FA)=0.$$

To prove the converse, i.e. $E_m^o\subseteq H_m$, let $F\in E_m^o$. It is desired to show $F_{ij}=0\ \forall i,j\leq m$. But if $i,j\leq m$ then $E_{ij}\in E_m$ and $\hat{E}_{ij}\in E_m$. Also, $Tr(FE_{ij}+E_{ij}F)=4f_{ij}$ and $Tr(F\hat{E}_{ij}+\hat{E}_{ij}F)4\hat{f}_{ij}$, by Lemma (4.2). But these must be zeros since $F\in E_m^o$.

LEMMA 4.4. *Let* L *be a subspace of* E *and let* U *be a unitary matrix. Then* $(\bar{U}^TLU)^o=\bar{U}^TL^oU$

Proof. Let $F\in(\bar{U}^TLU)^o$ then $Tr(F\bar{U}^TAU+\bar{U}^TAUF)=0,\ \forall\ A\in L$. Hence $Tr(\bar{U}^T(UF\bar{U}^TA)U+\bar{U}^T(AUF\bar{U}^T)U)=Tr(UF\bar{U}^TA+AUF\bar{U}^T)=0$. Therefore, $UF\bar{U}^T\in L^o$ and so $F\in\bar{U}^TL^oU$.

Conversely, let $F\in\bar{U}^TL^oU$ then $UF\bar{U}^T\in L^o$. Hence by a similar computation as above, $F\in(\bar{U}^TLU)^o$.

LEMMA 4.5. *Let* L *be a perfect subspace of* E *and* U *be a unitary matrix. Then* $\bar{U}^TLU$ *is a perfect subspace.*

The trivial proof of this is omitted.

LEMMA 4.6. *Given* $\varepsilon>0$, *there exists* $\lambda_\varepsilon>0$ *such that*

$$\sum_{k=1}^{m}|x_k|\leq\lambda_\varepsilon\sum_{k=1}^{m}|x_k|^2+\varepsilon \quad \textit{for all } (x_1,x_2,\ldots x_m)\in C^m \textit{ with } \sum_{k=1}^{m}|x_k|\leq 1.$$

Proof. Choose $\lambda_\varepsilon>\frac{m^2}{\varepsilon^2}$. Then for $|x_k|<\frac{\varepsilon}{m}$, $k=1,2,\ldots m$, $\sum_{k=1}^{m}|x_k|<\varepsilon$. Hence $\sum_{k=1}^{m}|x_k|<\varepsilon+\lambda_\varepsilon\sum_{k=1}^{m}|x_k|^2$. If $|x_l|\geq\frac{\varepsilon}{m}$ for some l, then

$$\lambda_\varepsilon \sum_{k=1}^{m} |x_k|^2 \geq \lambda_\varepsilon \cdot |x_l|^2 \geq \frac{m^2}{\varepsilon^2} \cdot \left(\frac{\varepsilon}{m}\right)^2 = 1 \geq \sum_{m=1}^{m} |x_k|$$

LEMMA 4.7. *Let* $H_m = \{A \in E \mid A_{ij} = 0 \forall i, j \leq m\}$, $m < n$ *and* L *be a subspace of* H_m *such that* $E_{kk} \in L$ $\forall k > m$. *Then* L *is a perfect subspace.*

Proof. Let $A \in L$ and let $M = \max \{|a_{ij}| \mid i \geq m+1, \ j = 1, 2, \ldots n\}$. Choose N such that $N > M(m+n)\dfrac{(n-m)^2}{\varepsilon^2}$ and define a diagonal matrix B such that $B = N\sum_{k=m+1}^{n} E_{kk}$. We claim that A, $0 \leq B + \varepsilon$.

To prove the claim, take an arbitrary vector X with $\sum_{i=1}^{n} |x_i| = 1$. Then

$$\bar{X}^T AX = \sum_{i=1}^{n} \bar{x}_i \sum_{j=1}^{n} a_{ij} x_j = \sum_{i,j} a_{ij} \bar{x}_i x_j,$$

and hence

$$\begin{aligned}
|\bar{X}^T AX| &\leq \sum_{i,j} |a_{ij}|\,|x_i|\,|x_j| = \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|\,|x_i|\,|x_j| + \sum_{i=m+1}^{n} \sum_{j=1}^{n} |a_{ij}|\,|x_i|\,|x_j| \\
&\leq M \sum_{i=1}^{m} \sum_{j=m+1} |x_i|\,|x_j| + M \sum_{i=m+1}^{n} \sum_{i=1}^{n} |x_i|\,|x_j| \\
&\leq M \sum_{i=1}^{m} \sum_{j=m+1}^{n} |x_j| + M \sum_{i=m+1}^{n} \sum_{j=1}^{n} |x_i| \leq M(m+n) \sum_{k=m+1}^{n} |x_k|
\end{aligned}$$

Now, by Lemma (4.6),

$$\sum_{k=m+1}^{m} |x_k| \leq \frac{(n-m)^2}{\varepsilon^2} \sum_{k=m+1}^{n} |x_k|^2 + \varepsilon$$

Therefore, $|\bar{X}^T AX| \leq M(n+m)$.

$$\begin{aligned}
\frac{(n-m)^2}{\varepsilon^2} \sum_{k=m+1}^{n} |x_k|^2 + \varepsilon &< N \sum_{k=m+1}^{n} |X_k|^2 + \varepsilon \\
&= \bar{X}^T BX + \varepsilon
\end{aligned}$$

THEOREM 4.8. *Let* L *be a subspace of* E. *Then* L *is a perfect subspace if and only if there exists* $m \leqslant n$ *such that* $L = \bar{U}^T L_1 U$ *for some unitary* U *and a subspace* L_1 *of* H_m *with* $E_{kk} \in L_1 \forall k \geq m$

Proof. If L is not a proper subspace, there is nothing to prove. Hence, we assume L is a proper subspace. If part of the theorem follows from Lemma (4.6) and (4.7). To prove the only if part, note that since L^o is an order ideal $L^o = \bar{U}^T (E_m + L_2) U$ for some unitary matrix U and a subspace L_2 of H_{mo} by Theorem (3.8). Thus,

$$E_m + L_2 = UL^o \bar{U}^T$$

and hence

$$(E_m + L_2)^o = (UL^o \bar{U}^T)^o = UL^{oo} \bar{U}^T = UL\bar{U}^T$$

by Lemma (4.4), and by the fact that L is a finite dimensional subspace which is always closed. From this, we obtain

$$L = \bar{U}^T (E_m + L_2)^o U$$

Now, let $L_1 = (E_m + L_2)^o$. Then clearly, L_1 is a subspace of H_m since so is $E_m{}^o$. It is left to show $E_{kk} \in L_1$ $\forall k > m$. But, by Lemma (4.2), $Tr(E_{kk}A + AE_{kk}) = 4A_{kk}$ and if $A \in E_m + L_2$ with $L_2 \subseteq H_{mo}$ then $a_{kk} = 0$ $\forall k \geq m+1$. Thus, $E_k \in L_1$.

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