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PROLONGATION OF VECTOR-VALUED DIFFERENTIAL FORMS TO THE FRAME BUNDLE

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Introduction

The purpose of this paper is to construct the prolongation of a vector-valued differential form on a manifold M to the frame bundle FM of M. Our goal is to get a general framework in which the theory of prolongations to FM of tensor fields, G-structures and connections on M, developed in [1, 2, 3, 6], falls in a natural way.

To do this, we proceed as follows. If ω is a V-valued form on M, V a vector space, a $J_p{}^1V$-valued form ω_1 can be constructed on $J_p{}^1M$ ($J_p{}^1$ denotes the tangent bundle of p^1-velocities); moreover, if ω is of type (ρ, V), ρ being a linear representation of a Lie group G (acting on M) on V, then ω_1 is shown to be of type $(\rho_1, J_p{}^1V)$, ρ_1 being an induced representation of $J_p{}^1G$ on $J_p{}^1V$. The particular case of forms taking values in a Lie algebra is considered in Section 4. Applications to the prolongations of connections are made in Section 5. Sections 6 and 7 are devoted to particularize the general results for appropriate vector spaces, in order to obtain geometrical interpretations of the definitions and results in [1, 2, 6] concerning the prolongation to FM of tensor fields and G-structures on M.

Through the paper, manifolds, maps, tensor fields and so on, will be assumed differentiable of class C^∞, and the manifolds to be connected. Summation over repeated indices is always implied; entries of matrices are written as $a^i{}_j, a_{ij}$ or a^{ij} and in all cases, i is the row index while j is the column index. $Gl(n)=Gl(n, R)$ is the general linear group and $gl(n)=gl(n, R)$ is the Lie algebra of all $n\times n$ square matrices.

1. Generalities on $J_p{}^1M$

Let R^p be the Euclidean p-space, M an n-dimensional manifold,

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$J_p{}^1M$ the $(n+pn)$-dimensional manifold of 1-jets j^1f at $0\in R^p$ of differentiable mappings $f: R^p \longrightarrow M$ defined on some open neighborhood of 0, and $\pi_M: J_p{}^1M \longrightarrow M$ the target map, $\pi_M(j^1f)=f(0)$.

If (U, x^i) is a coordinate system in M, then the induced coordinates $(x^i, x_\alpha{}^i)$ on $J_p{}^1\ U=\pi_M{}^{-1}(U)$ are given by

$$x^i(j^1f)=x^i(f(0)),\quad x_\alpha{}^i(j^1f)=\frac{\partial(x^i\cdot f)}{\partial t^\alpha}(0),\quad 1\leqslant\alpha\leqslant p,\quad 1\leqslant i\leqslant n$$

for any $j^1f\in J_p{}^1U$, where $(t^1, \ldots, t^p)$ are the usual coordinates in R^p.

Let $h: M \longrightarrow M'$ be a differentiable map; then, the induced map $h^1: J_p{}^1M \longrightarrow J_p{}^1M'$ is given by $h^1(j^1f)=j^1(h\cdot f)$. If (U, x^i), (U', y^j) are coordinate systems in M and M', respectively, and if we suppose $h: U \longrightarrow U'$ expressed by $y^j=h^j(x^i)$, then, with respect to $(J_p{}^1U, x^i, x_\alpha{}^i)$, $(J_p{}^1U', y^j, y_\alpha{}^j)$, h^1 is expressed by

$$h^1:\ y^j=h^j(x^i),\quad y_\alpha{}^j=\frac{\partial h^j}{\partial x^i}x_\alpha{}^i \tag{1.1}$$

Obviously $\pi_{M'}h^1=h\cdot\pi_M$, $(1_M)^1=1_{J_p{}^1M}$ and $(k\cdot h)^1=k^1\cdot h^1$ if $k: M' \longrightarrow M''$. Moreover, there exists a canonical diffeomorphism $J_p{}^1(M\times N)\simeq J_p{}^1M\times J_p{}^1N$ for any differentiable manifolds M, N, and hereafter we shall identify them; also, if $h_i: M_i \longrightarrow N_i$, $i=1, 2$, then $(h_1\times h_2)^1 =h_1{}^1\times h_2{}^1$.

Take $p=n=\dim M$. Then $J_n{}^1M$ contains as an open (dense) submanifold the bundle space FM of the principal bundle of linear frames over M (briefly, the frame bundle of M) and, in fact, the manifold sructure which $J_n{}^1M$ induces on FM is the usual one with respect to which $\pi_M: FM \longrightarrow M$ is a $Gl(n)$-principal bundle. Through this paper, the coordinates induced on $FU=J_n{}^1U\cap FM$ will be written $(x^i, X^i{}_j)$ if there is no confusion.

If $p=1$, then $\pi_M: J_1{}^1M \longrightarrow M$ is nothing but the tangent bundle of M, $\pi_M: TM \longrightarrow M$, and the coordinates induced on $TU=\pi_M{}^{-1}(U)$ will be written $(x^i; \dot{x}^i)$; $Th: TM \longrightarrow TN$ will denote the map induced by $h: M \longrightarrow N$, and, for any $f: R \longrightarrow M$, j^1f will be written simply as $\dot{f}$.

Particularizing some general results of Morimoto [7], we can assert: there exist canonical diffeomorphisms

$$\alpha_M^{p,1}: TJ_p{}^1M \longrightarrow J_p{}^1TM,\quad \alpha_M^{1,p}J_p{}^1TM \longrightarrow TJ_p{}^1M$$

such that both are mutually inverse. For further use, we shall recall their definition.

Let $\dot{\varphi}\in T_{\varphi(0)}J_p{}^1M$ be the tangent vector to $\varphi : R \longrightarrow J_p{}^1M$ at $\varphi(0)$; then, there exist a differentiable map $\psi : R\times R^p \longrightarrow M$ and a positive number δ such that $\varphi(t)=j^1\psi_t$ for $|t|<\delta$, where $\psi_t(u)=\psi^u(t)=\psi(t, u)$, $t\in R$, $u\in R^p$. Define $\Psi : R^p \longrightarrow TM$ by $\Psi(u)=\dot{\psi}^u$; then $\alpha_M{}^{p,1}(\dot{\varphi})=j^1\Psi$.

Analogously, let $\eta : R^p \longrightarrow TM$ be a differentiable map; then, there exist a differentiable map $\phi : R^p\times R \longrightarrow M$ and a positive number δ such that $\eta(t)=\dot{\phi}_t$ for $|t|<\delta$, where $\phi_t(u)=\phi^u(t)=\phi(t, u)$, $t\in R^p$, $u\in R$. Define $\Phi : R \longrightarrow J_p{}^1M$ by $\Phi(u) =j^1\phi^u$; then $\alpha_M^{1,p}(j^1\eta)=\dot{\Phi}$.

Locally, $\alpha_M^{p,1}$ is given as follows: let (U, x^i) be a coordinate system in M, and $(x^i, x_\alpha{}^i; \dot{x}^i, \dot{x}_\alpha{}^i)$, $(y^i, \dot{y}^i, (y^i)_\alpha, (\dot{y}^i)_\alpha)$ the induced coordinates on $TJ_p{}^1U$ and $J_p{}^1TU$, respectively; then

$$\alpha_M^{p,1} : y^i=x^i,\quad \dot{y}^i=\dot{x}^i,\quad (y^i)_\alpha=x_\alpha{}^i,\quad (\dot{y}^i)_\alpha=\dot{x}_\alpha{}^i \tag{1.2}$$

The local expression for $\alpha_M^{1,p}$ is obvious. Moreover, for any $f : M\longrightarrow N_0$,

$$(Tf)^1\cdot\alpha_M^{p,1}=\alpha_N^{p,1}\cdot Tf^1,\quad Tf^1\cdot\alpha_M^{1,p}=\alpha_N^{1,p}\cdot(Tf)^1 \tag{1.3}$$

Let $X : M \longrightarrow TM$ be a vector field on M, i.e. $\pi_M\cdot X=1_M$. Then

$$X^C=\alpha_M^{1,p}\cdot X^1 : J_p{}^1M \longrightarrow TJ_p{}^1M$$

is a cross-section of the tangent bundle $TJ_p{}^1M$ and, hence, it defines a vector field X^C on $J_p{}^1M$ which will be said the *complete lift* of X to $J_p{}^1M$. If X is locally given in U by $X=X^i(\partial/\partial x^i)$, then X^C is given on $J_p{}^1U$ by

$$X^C=X^i\frac{\partial}{\partial x^i}+x_\alpha{}^j\frac{\partial X^i}{\partial x^j}\frac{\partial}{\partial x_\alpha{}^i}$$

Moreover, for each $\alpha=1, 2, \ldots, p$, there exists on $J_p{}^1M$ *a* vector field $X^{(\alpha)}$ associated to X, locally given by

$$X^{(\alpha)}=X^i\frac{\partial}{\partial x_\alpha{}^i}$$

and which will be said the *α^{th}-vertical lift* of X to $J_p{}^1M$. Since locally

$$\left[\frac{\partial}{\partial x^i}\right]^C=\frac{\partial}{\partial x^i},\quad \left[\frac{\partial}{\partial x^i}\right]^{(\alpha)}=\frac{\partial}{\partial x_\alpha{}^i}$$

any differential form on $J_p{}^1M$ is completely determined by its action on these lifts $X, X^{(\alpha)}$ for every vector field X on M. Moreover, the following identities hold:

$$[X^C, Y^C]=[X, Y]^C,\quad [X^C, Y^{(\alpha)}]=[X, Y]^{(\alpha)},\quad [X^{(\alpha)}, Y^{(\beta)}]=0$$

Let G be a Lie group with Lie algebra $\mathfrak{g}$; then $J_p{}^1G$ inherits a Lie group structure, its multiplication being defined by $(j^1f)\cdot(j^1g)=j^1(f\cdot g)$ for any $j^1f, j^1g\in J_p{}^1G$, where $(f\cdot g)(t)=f(t)g(t)$, $t\in R^p$. The following result has been proved in [2]: let $ad: G\longrightarrow Aut(\mathfrak{g})$, $Ad:\mathfrak{g}\longrightarrow Der(\mathfrak{g})$ be the adjoint representations, $\mathfrak{g}^p=\mathfrak{g}\times\overset{p}{\cdots}\times\mathfrak{g}$ considered as abelian Lie group, and construct the semidirect product Lie group $G\times_{ad}\mathfrak{g}^p$ and the semidirect product Lie algeara $\mathfrak{g}\times_{Ad}\mathfrak{g}^p$; then there is a canonical isomorphism of Lie groups $J_p{}^1G\simeq G\times_{ad}\mathfrak{g}^p$ and, consequently, the Lie algebra of $J_p{}^1G$ is isomorphic to $\mathfrak{g}\times_{Ad}\mathfrak{g}^p$.

Let be $G=Gl(n)$, $\{X_j{}^i\}$ the canonical coordinates in $Gl(n)$, $\{X_j{}^i, X_j{}^i{}_\alpha\}$ the induced coordinates in $J_p{}^1Gl(n)$, and $\{y_B{}^A,\ 1\leqslant A,\ B\leqslant n+pn\}$ the canonical coordinates in $Gl(n+pn)$. Then, there exists a canonical embedding of Lie groups [2]

$$j_p: J_p{}^1Gl(n)\longrightarrow Gl(n+pn)$$

given by

$$j_p((X_j{}^i,\ X_{j\alpha}{}^i))=\begin{pmatrix}(X_j{}^i) & 0\dots\dots\dots 0\\ (X_{j1}{}^i) & (X_j{}^i)\dots\dots 0\\ \vdots & \vdots\qquad\quad \vdots\\ (X_{jp}{}^i) & 0\dots(X_j{}^i)\end{pmatrix} \tag{1.4}$$

and the induced homomorphism $Tj_p: T_{\tilde{e}}J_p{}^1Gl(n)\longrightarrow T_eGl(n+pn)$ at the unit elements writes

$$Tj_p((\delta_j{}^i,\ 0; A_j{}^i,\ B_{j\alpha}{}^i))=(\delta_B{}^A;\begin{pmatrix}(A_j{}^i) & 0\dots\dots\dots 0\\ (B_{j1}{}^i) & (A_j{}^i)\dots\dots 0\\ \vdots & \vdots\qquad\quad \vdots\\ (B_{jp}{}^i) & 0\dots(A_j{}^i)\end{pmatrix})$$

Let $P(M,\pi,G)$ be a principal fibre bundle with bundle space P, base space M, projection π and structure group G. Then $J_p{}^1P(J_p{}^1M, \pi^1, J_p{}^1G)$ is again a principal fibre bundle. In fact, if $\phi_U:\pi^{-1}(U)\longrightarrow U\times G$ is a trivialization of P over $U\in M$, then, since $(\pi^1)^{-1}(J_p{}^1U)=J_p{}^1\pi^{-1}(U)$, an induced trivialization $\tilde{\phi}_U: J_p{}^1\pi^{-1}(U)\longrightarrow J_p{}^1U\times J_p{}^1G$ is obtained by setting $\tilde{\phi}_U(j^1f)=(j^1(\pi\cdot f), j^1(pr\cdot\phi_U\cdot f))$ for any $j^1f\in J_p{}^1\pi^{-1}(U)$, where $pr: U\times G\longrightarrow G$ is the canonical projection.

Let $FM(M,\ \pi_M,\ Gl(n))$ be the frame bundle of M, $J_n{}^1FM(J_n{}^1M, \pi_M{}^1,\ J_n{}^1Gl(n))$ the induced principal bundle and $FJ_n{}^1M(J_n{}^1M, \pi_{J_n{}^1M}, Gl(n+n^2))$ the frame bundle of $J_n{}^1M$. Then, there exists a canonical injective homomorphism of principal bundles over the identity of $J_n{}^1M$ [2]

$$j_M: J_n{}^1FM\longrightarrow FJ_n{}^1M$$

with associate Lie group homomorphism $j_n : J_n^1 Gl(n) \longrightarrow Gl(n+n^2)$; j_M is locally given as follows: let (U, x^i) be a coordinate system in M, and consider fibered coordinates $(x^i, x_\alpha^i, X_j^i, X_{j\alpha}^i)$ on $J_n^1 \pi_M^{-1}(U)$ and (y^i, y_α^i, Y_B^A) on $FJ_n^1 U (1 \leqslant A,\ B \leqslant n+n^2)$; then

$$j_M: \begin{array}{l} y^i = x^i, \ y_\alpha^i = x_\alpha^i \\ y_j^i = X_j^i, \ y_j^{i_\alpha} = X_{j\alpha}^i, \ y_{j\alpha}^i = 0, \ y_{j\beta}^{i_\alpha} = \delta_\beta^\alpha X_j^i \end{array} \tag{1.5}$$

where $i_\alpha = \alpha n + i$, $1 \leqslant \alpha \leqslant n$.

Since the restriction $FJ_n^1 M|_{FM}$ is isomorphic to the frame bundle FFM of FM, then j_M induces again $j_M : J_n^1 FM|_{FM} \longrightarrow FFM$ over the identity of FM.

Let G be a Lie subgroup of $Gl(n)$, $\tilde{G} = j_n(J_n^1 G) \subset Gl(n+n^2)$ and $P(M, \pi, G)$ a G-structure on M. In [2], we have defined the prolongation $\tilde{P}$ of P to FM as the $\tilde{G}$-structure on FM given as follows: let $j^1 : J_n^1 P \longrightarrow J_n^1 FM$ be induced by $j : P \longrightarrow FM$, then $\tilde{P} = (j_M \cdot j^1)(J_n^1 P)|_{FM}$.

2. $J_p^1 V$ for a vector space V

Let V be a (real) vector space, dim $V=m$. Fix, once and for all, a basis $\{e_a,\ 1 \leqslant a \leqslant m\}$ of V and consider V as m-dimensional manifold. Then, $J_p^1 V$ inherits a vector space structure: for any $j^1 f,\ j^1 g \in J_p^1 V$ and $\lambda \in R$, define

$$j^1 f + j^1 g = j^1(f+g), \quad \lambda(j^1 f) = j^1(\lambda f)$$

An induced basis of $J_p^1 V$ is constructed as follows: define $f_a, f_{a_\alpha} : R^p \longrightarrow V$ by

$$f_a(t) = e_a, \quad f_{a_\alpha}(t) = t^\alpha e_a, \quad t = (t^1, \dots, t^p) \in R^p$$

and set $E_a = j^1 f_a$, $E_{a_\alpha} = j^1 f_{a_\alpha}$; then, $\{E_a,\ E_{a_\alpha}\}$ is a basis for $J_p^1 V$.

On the other hand, the vector space $V^{1+p} = V \times V^p$ has a canonical basis $\{E_a', E_{a_\alpha}'\}$ induced from $\{e_a\}$: $E'_a = (e_a, 0, \dots, 0)$, $E'_{a_\alpha} = (0, \dots, e_a, \dots, 0)$ with e_a at the $(\alpha+1)^{th}$ place, and the correspondence $E_a \longrightarrow E'_a$, $E_{a_\alpha} \longrightarrow E'_{a_\alpha}$ defines an isomorphism of vector spaces $J_p^1 V \simeq V^{1+p}$; this isomorphism being, in fact, a diffeomorphism, hereafter $J_p^1 V$ and V^{1+p} will be identified without explicit mention.

Assume $V = \mathfrak{g}$ a (real) Lie algebra; then $J_p^1 \mathfrak{g}$ inherits a Lie algebra structure; define $[j^1 f,\ j^1 g] = j^1[f,\ g]$, where $[f, g](t) = [f(t), g(t)]$, $t \in R^p$. In fact, if $\{\lambda_{ab}^c\}$ are the structure constants of $\mathfrak{g}$ with respect to $\{e_a\}$, i.e. $[e_a,\ e_b] = \lambda_{ab}^c e_c$, then the structure constants of $J_p^1 \mathfrak{g}$ with respect to $\{E_a, E_{a_\alpha}\}$ are

$$\begin{aligned} &\Lambda_{ab}{}^c=\lambda_{ab}{}^c, \quad \Lambda_{ab_\beta}{}^{c_\alpha}=\delta_\beta{}^\alpha\lambda_{ab}{}^c \\ &\Lambda_{ab_\alpha}{}^c=\Lambda_{ab}{}^{c_\alpha}=\Lambda_{a_\beta}{}^c{}_{b_\gamma}=\Lambda_{a_\beta}{}^{c_\alpha}{}_{b_\gamma}=0 \end{aligned} \tag{2.1}$$

Let $Ad : \mathfrak{g} \longrightarrow Der\ (\mathfrak{g}^p)$ be the canonical extension of the adjoint representation of $\mathfrak{g}$ to the abelian Lie algebra $\mathfrak{g}^p$, and construct the semidirect product Lie algebra $\mathfrak{g}\times_{Ad}\mathfrak{g}^p$; then

LEMMA 2.1. *The isomorphism of vector spaces $J_p{}^1\mathfrak{g}\simeq\mathfrak{g}\times_{Ad}\mathfrak{g}^p$ is, in fact, an isomorphism of Lie algebras.*

Proof. Routine, having in mind (2.1) and the definition of the bracket product in $\mathfrak{g}\times_{Ad}\mathfrak{g}^p$.

Let V be again a vector space, and consider the Lie groups $J_p{}^1Gl(V)$ and $Gl(J_p{}^1V)$; we define a differentiable embedding of Lie groups $j_p : J_p{}^1Gl(V) \longrightarrow Gl(J_p{}^1V)$ by setting

$$j^p(j^1f)(j^1\eta)=j^1(f*\eta), \quad j^1f\in J_p{}^1Gl(V), \quad j^1\eta\in J_p{}^1V \tag{2.2}$$

where $f*\eta : R^p \longrightarrow V$ is given by $(f*\eta)(t)=f(t)((\eta)(t))$. In particular, when $V=R^n$ we refind the homomorphism j_p in (1.4).

Let $\rho : G \longrightarrow Gl(V)$ be a linear representation of a Lie group G into V; then

$$\rho_1=j_p\cdot\rho^1 : J_p{}^1G \longrightarrow Gl(J_p{}^1V)$$

is again a linear representation which will be said induced by ρ. In fact

$$\rho_1(j^1\eta)=j_p(j^1(\rho\cdot\eta)), \quad j^1\eta\in J_p{}^1G \tag{2.3}$$

Let $\phi : G\times V \longrightarrow V$ denote the action of G on V induced by ρ, and let $\phi^1 : J_p{}^1G\times J_p{}^1V \longrightarrow J_p{}^1V$ be the induced map.

LEMMA 2.2. *ϕ^1 is the action induced by ρ_1.*

Proof. Let $\lambda : Gl(V)\times V \longrightarrow V$, $\tilde{\lambda} : Gl(J_p{}^1V)\times J_p{}^1V \longrightarrow J_p{}^1V$ be the natural actions. Then $\phi=\lambda\cdot(\rho\times 1_V)$ and $\tilde{\phi}=\tilde{\lambda}\cdot(\rho_1\times 1_{J_p{}^1V})$ are the actions induced by ρ and ρ_1, respectively. Since $\lambda^1=\tilde{\lambda}\cdot(j_p\times 1_{J_p{}^1V})$,

$$\begin{aligned} \phi^1&=(\lambda\cdot(\rho\times 1_V))^1=\lambda^1\cdot(\rho^1\times 1_{J_p{}^1V})=\tilde{\lambda}\cdot((j_p\cdot\rho^1)\times 1_{J_p{}^1V}) \\ &=\tilde{\lambda}\cdot(\rho_1\times 1_{J_p{}^1V})=\tilde{\phi} \end{aligned}$$

The following Lemma will be useful later.

LEMMA 2.3. *Let $\eta : R^p \longrightarrow G$ be a differentiable map, $\tilde{g}=j^1\eta\in J_p{}^1G$, and $f : R^p \longrightarrow TV$ such that $f(R^p)\subset T_0V$, the tangent space to V at*

0. *Define* $T\rho(\eta^{-1})*f : R^p \longrightarrow TV$ *by*

$$(T\rho(\eta^{-1})*f)(t) = T\rho(\eta^{-1}(t))(f(t))$$

Then:

(i) $\alpha_V^{1,p}(j^1 f) \in T_0 J_p{}^1 V$

(ii) $\alpha_V^{1,p}(j^1(T\rho(\eta^{-1})*f)) = T\rho_1(\tilde{g}^{-1})(\alpha_V^{1,p}(j^1 f))$

Proof. Let $\sigma : R^p \times R \longrightarrow V$ be such that $f(t) = \dot{\sigma}_t$ for sufficiently small $t \in R^p$, where $\sigma_t(u) = \sigma^u(t) = \sigma(t, u)$, $u \in R$. Since $f(t) \in T_0 V$ for every t, then $\sigma_t(0) = 0 \in V$; consequently, $\sigma^0(t) = \sigma_t(0) = 0$, that is $\sigma^0 : R^p \longrightarrow V$ is the constant map into $0 \in V$, and therefore $j^1\sigma^0 = 0 \in J_p{}^1 V$. But $\alpha_V^{1,p}(j^1 f) = \dot{\Sigma}$, $\Sigma : R \longrightarrow J_p{}^1 V$ being given by $\Sigma(u) = j^1\sigma^u$, and then $\Sigma(0) = j^1\sigma^0$ and (i) is proved.

Now, for small t, $(T\rho(\eta^{-1})*f)(t) = \dot{\sigma}_t{}'$, where $\sigma_t{}'(u) = \rho(\eta^{-1}(t))(\sigma_t(u))$; then, if we define $\Sigma' : R \longrightarrow J_p{}^1 V$ *by* $\Sigma'(u) = j^1\sigma'^u$, we get $\alpha_V^{1,p}(j^1(T\rho(\eta^{-1})*f)) = \dot{\Sigma}'$. On the other hand, let $\rho(\eta^{-1})*\sigma^u : R^p \longrightarrow V$ be given by $(\rho(\eta^{-1})*\sigma^u)(t) = \rho(\eta^{-1}(t))(\sigma^u(t))$; then, by virtue of (2.3),

$$\Sigma'(u) = j^1(\rho(\eta^{-1})*\sigma^u) = j_p(j^1(\rho\cdot\eta^{-1}))(j^1\sigma^u) = \rho_1(\tilde{g}^{-1})(j^1\sigma^u)$$

and (ii) is proved.

Also, remark that if $E(M, \pi, V)$ is a vector bundle with standard fibre V, then $J_p{}^1 E(J_p{}^1 M, \pi^1, J_p{}^1 V)$ is again a vector bundle with standard fibre $J_p{}^1 V$. In fact, this corresponrdence preserves Whitney sums, that is, if $E'(M, \pi', V')$ is another vector bundle, then $J_p{}^1(E \oplus E')$ and $J_p{}^1 E \oplus J_p{}^1 E'$ are isomorphic vector bundles; in particular, for any integer $r \geqslant 2$, $J_p{}^1(\oplus_r TM)$ and $\oplus_r(J_p{}^1 TM)$ are isomorphic vector bundles.

Next, let us consider the following vector spaces:

$$J_p{}^1 R^n \simeq R^n \times (R^n)^p \equiv R^{n+pn}$$
$$\otimes_s{}^1 R^n = R^n \otimes (R^n)^* \otimes \overset{s}{\cdots} \otimes (R^n)^* \simeq L_s(R^n, R^n)$$
$$\otimes_s R^n = (R^n)^* \otimes \overset{s}{\cdots} \otimes (R^n)^* \simeq L_s(R^n, R)$$

where $L_s(V, W)$ is the vector space of s-linear maps of $V \times \overset{s}{\cdots} \times V$ into W; define $\pi : J_p{}^1 R \longrightarrow R$ by

$$\pi(t, c_1, \dots, c_p) = \sum_{\alpha=1}^{p} c_\alpha \tag{2.4}$$

and denote by $\{e^i\}$ the basis of $(R^n)^*$ dual to the canonical basis $\{e_i\}$ of R^n, $\{E^i, E_\alpha{}^i\}$ the basis of $(J_p{}^1 R^n)^*$ dual to the induced basis $\{E_i, E_{i_\alpha}\}$ of $J_p{}^1 R^n \equiv R^{n+pn}$, and by $\{e_i \otimes e^{j_1} \otimes \dots \otimes e^{j_s}, (e_i \otimes e^{j_1} \otimes \dots \otimes e^{j_s})_\alpha\}$ and

$\{e^{j1}\otimes\ldots\otimes e^{js}, (e^{j1}\otimes\ldots\otimes e^{js})_\alpha\}$ the induced basis of $J_p{}^1(\otimes_s{}^1R^n)$ and $J_p{}^1(\otimes_s R^n)$, respectively.

Then, the following homomorphisms of vector spaces can be defined:

(I) $\qquad i : J_p{}^1(\otimes_s{}^1R^n) \longrightarrow \otimes_s{}^1(J_p{}^1R^n)\equiv\otimes_s{}^1R^{n+pn}$

Let be $j^1f\in J_p{}^1(\otimes_s{}^1R^n)$ with $f : R^p \longrightarrow \otimes_s{}^1R^n$; thus, $f(t)\in L_s(R^n, R^n)$ for each t. We define $i(j^1f)\in L_s(J_p{}^1R^n, J_p{}^1R^n)$ as follows: for any $j^1g_1, \ldots, j^1g_s\in J_p{}^1R^n$,

$$i(j^1f)(j^1g_1, \ldots, j^1g_s)=j^1(f*(g_1, \ldots, g_s))$$

where $(f*(g_1, \ldots, g_s))(t)=f(t)(g_1(t), \ldots, g_s(t))$. Then, a straightforward computation from the definitions leads to:

$$\begin{aligned} i(e_i\otimes e^{j1}\otimes\cdots\otimes e^{js}) &= E_i\otimes E^{j1}\otimes\cdots\otimes E^{js}+ \\ &+\sum_{k=1}^{s}\sum_{\alpha=1}^{p} E_{i_\alpha}\otimes E^{j1}\otimes\cdots\otimes E_\alpha{}^{jk}\otimes\cdots\otimes E^{js} \\ i((e_i\otimes e^{j1}\otimes\cdots\otimes e^{js})_\alpha) &= E_{i_\alpha}\otimes E^{j1}\otimes\cdots\otimes E^{js} \end{aligned} \tag{2.5}$$

(II) $\qquad i : J_p{}^1(\otimes_s R^n) \longrightarrow \otimes_s(J_p{}^1R^n)\equiv\otimes_s R^{n+pn}$

Let be $j^1f\in J_p{}^1(\otimes_s R^n)$ with $f : R^p \longrightarrow \otimes_s R^n$; thus, $f(t)\in L_s(R^n, R)$ for each t. We define $i(j^1f)\in L_s(J_p{}^1R^n, R)$ as follows: for any $j^1g_1, \ldots, j^1g_s\in J_p{}^1R^n$,

$$i(j^1f)(j^1g_1, \ldots, j^1g_s)=\pi(j^1\langle f, (g_1, \ldots, g_s)\rangle)$$

where $\langle f, (g_1, \ldots, g_s)\rangle(t)=f(t)(g_1(t), \ldots, g_s(t))$. Then, a straightforward computation from the definitions leads to

$$\begin{aligned} i(e^{j1}\otimes\cdots\otimes e^{js}) &= \sum_{k=1}^{s}\sum_{\alpha=1}^{p} E^{j1}\otimes\cdots\otimes E_\alpha{}^{jk}\otimes\cdots\otimes E^{js} \\ i((e^{j1}\otimes\cdots\otimes e^{js})_\alpha) &= E^{j1}\otimes\cdots\otimes E^{js} \end{aligned} \tag{2.6}$$

Both homomorphisms i above behave appropriately with respect to canonical representations; that is, let be $\rho : Gl(n) \longrightarrow Gl(\otimes_s{}^1R^n)$ or $\rho : Gl(n) \longrightarrow Gl(\otimes_s R^n)$ the canonical linear representations given by

$$(\rho(\eta)f)(\xi_1, \ldots, \xi_s)=\eta(f(\eta^{-1}\xi_1, \ldots, \eta^{-1}\xi_s)), \ f\in\otimes_s{}^1R^n$$

or

$$(\rho(\eta)f)(\xi_1, \ldots, \xi_s)=f(\eta\xi_1, \ldots, \eta\xi_s), \ f\in\otimes_s R^n$$

for any $\eta\in Gl(n)$, $\xi_1, \ldots, \xi_s\in R^n$; denote by $\tilde\rho : Gl(n+pn) \longrightarrow Gl(\otimes_s{}^1 R^{n+pn})$ or $\tilde\rho : Gl(n+pn) \longrightarrow Gl\otimes_s R^{n+pn})$ the analogous ones, and let $\rho_1 : J_p{}^1Gl(n) \longrightarrow Gl(J_p{}^1(\otimes_s{}^1R^n))$ or $\rho_1 : J_p{}^1Gl(n) \longrightarrow Gl(J_p{}^1(\otimes_s R^n))$ be the induced representations. Then, for any $\tilde g\in J_p{}^1Gl(n)$,

$$i\cdot\rho_1(\tilde g)=\tilde\rho(j_p(\tilde g))\cdot i \tag{2.7}$$

In fact, let be $\eta: R^p \longrightarrow Gl(n)$ such that $\tilde{g}=j^1\eta$; then, for any $j^1f\in J_p{}^1(\otimes_s{}^1R^n)$,

$$\rho_1(\tilde{g})(j^1f)=j_p(j^1(\rho\cdot\eta))(j^1f)=j^1((\rho\cdot\eta)*f)$$

where $((\rho\cdot\eta)*f)(t)=\rho(\eta(t))(f(t))$. Therefore, for any $j^1g_1, \ldots, j^1g_s\in J_p{}^1R^n$,

$$i(\rho_1(\tilde{g})(j^1f))(j^1g_1, \ldots, j^1g_s)=j^1(((\rho\cdot\eta)*f)*(g_1, \ldots, g_s))$$

where $(((\rho\cdot\eta)*f)*(g_1, \ldots, g_s))(t)=\eta(t)(f(t)(\eta(t)^{-1}g_1(t), \ldots, \eta(t)^{-1}g_s(t)))$. On the other hand,

$$\begin{aligned}
&\tilde{\rho}(j_p(\tilde{g}))(i(j^1f))(j^1g_1, \ldots, j^1g_s)\\
&=j_p(\tilde{g})(i(j^1f)(j_p(\tilde{g}^{-1})(j^1g_1), \ldots, j_p(\tilde{g}^{-1})(j^1g_s)))\\
&=j_p(\tilde{g})(i)j^1f)(j^1(\eta^{-1}*g_1), \ldots, j^1(\eta^{-1}*g_s))) \quad \text{(by (2.2))}\\
&=j_p(\tilde{g})(j^1(f*(\eta^{-1}*g_1, \ldots, \eta^{-1}*g_s)))\\
&=j^1(\eta*(f*(\eta^{-1}*g_1, \ldots, \eta^{-1}*g_s))) \quad \text{(by (2.2))}
\end{aligned}$$

and (2.7) holds for the type $(1, s)$. We omit the proof for the case $(0, s)$, which is similar.

3. Prolongation of vector-valued functions and forms

Let V be a vector space as in Section 2. Define injections i_0, $i_\alpha: V \longrightarrow J_p{}^1V$ by

$$i_0(\xi)=j^1f_\xi,\quad i_\alpha(\xi)=j^1f_{\alpha,\xi},\quad 1\leqslant\alpha\leqslant p,\quad \xi\in V$$

where f_ξ, $f_{\alpha,\xi}: R^p \longrightarrow V$ are given by $f_\xi(t)=\xi$, $f_{\alpha,\xi}(t)=t^\alpha\xi$, $t=(t^1, \ldots, t^p)\in R^p$.

Let $h: M \longrightarrow V$ be a V-valued differentiable function, $h^1: J_p{}^1M \longrightarrow J_p{}^1V$ the induced one, and define $h^{(\alpha)}: J_p{}^1M \longrightarrow J_p{}^1V$ by setting $h^{(\alpha)}=i_\alpha\cdot h\cdot\pi_M$ for each $\alpha\in\{0, 1, 2, \ldots, p\}$. Then, if h is locally expressed with respect to the basis $\{e_a\}$ by $h(x^i)=h^a(x^i)e_a$, the local expressions of h^1, $h^{(\alpha)}$ with respect to the induced basis $\{E_a, E_{a_\alpha}\}$ are

$$\begin{aligned}
&h^1(x^i,\ x_\alpha{}^i)=h^a(x^i)E_a+x_\alpha{}^j\frac{\partial h^a}{\partial x^j}E_{a_\alpha}\\
&h^{(0)}(x^i, x_\alpha{}^i)=h^a(x^i)E_a\\
&h^{(\alpha)}(x^i,\ x_\beta{}^i)=h^a(x^i)E_{a_\alpha},\quad 1\leqslant\alpha\leqslant p
\end{aligned}$$

Hence, for any vector field X on M,

$$\begin{aligned}
&X^ch^1=(Xh)^1,\quad X^ch^{(\alpha)}=(Xh)^{(\alpha)},\quad 0\leqslant\alpha\leqslant p\\
&X^{(\alpha)}h^1=(Xh)^{(\alpha)},\quad X^{(\alpha)}h^{(\beta)}=0,\quad 1\leqslant\alpha\leqslant p,\quad 0\leqslant\beta\leqslant p
\end{aligned}$$

If $V=\mathfrak{g}$ is a Lie algebra, then for any f, $g: M \longrightarrow \mathfrak{g}$ we denote

$[f, g] : M \longrightarrow \mathfrak{g}$ the bracket product map, and the following identities hold:

$$\begin{aligned}&[f, g]^1 = [f^1, g^1], \quad [f, g]^{(0)} = [f^{(0)}, g^{(0)}]\\ &[f, g]^{(\alpha)} = [f^{(\alpha)}, g^1] = [f^1, g^{(\alpha)}] = [f^{(0)}, g^{(\alpha)}] = [f^{(\alpha)}, g^{(0)}] \quad (3.1)\\ &[f^{(\alpha)}, g^{(\beta)}] = 0\end{aligned}$$

for $1 \leqslant \alpha, \beta \leqslant p$.

Next, let G be a Lie group acting on M on the right through $\psi : M\times G \longrightarrow M$, $\rho : G \longrightarrow Gl(V)$ a linear representation and $\phi : G\times V \longrightarrow V$ the action induced by ρ. For each $h : M \longrightarrow V$, let $i\times h : M\times G \longrightarrow G\times V$ be given by $(i\times h)(x, g) = (g^{-1}h(x))$.

DEFINITION 3.1. A differentiable map $h : M \longrightarrow V$ is said of type (ρ, V) if $h\cdot\psi = \phi\cdot(i\times h)$, or equivalently $h(xg) = \rho(g^{-1})(h(x))$ for any $x\in M$, $g\in G$.

$J_p{}^1G$ acts on $J_p{}^1M$ on the right through ψ^1, and ϕ^1 is the action of $J_p{}^1G$ on $J_p{}^1V$ induced by $\rho_1 : J_p{}^1G \longrightarrow Gl(J_p{}^1V)$; then, since $(i\times h)^1 = i\times h^1$, for any V-valued differentiable map on M of type (ρ, V) we have $h^1\cdot\psi^1 = (h\cdot\psi)^1 = (\phi\cdot(i\times h))^1 = \phi^1\cdot(i\times h^1)$, and thus the following theorem is proved:

THEOREM 3.2. *If* $h : M \longrightarrow V$ *is of type* (ρ, V), *then* $h^1 : J_p{}^1M \longrightarrow J_p{}^1V$ *is of type* $(\rho_1, J_p{}^1V)$.

Let ω be a V-valued 1-form on M; such a form ω can be considered canonically as a differentiable map $\omega : TM \longrightarrow TV$ which is a linear map of the tangent space T_xM with values in the tangent space T_0V for each $x\in M$.

So, let $\omega : TM \longrightarrow TV$ be a V-valued 1-form on M, and define a differentiable map $\omega_1 : TJ_p{}^1M \longrightarrow TJ_p{}^1V$ by setting

$$\omega_1 = \alpha_V{}^{1,p}\cdot\omega^1\cdot\alpha_M^{p,1} \tag{3.2}$$

where $\omega^1 : J_p{}^1TM \longrightarrow J_p{}^1TV$ is the map induced by ω. Let (U, x^i) be a coordinate system in M, $(TJ_p{}^1U, (x^i, x_\alpha{}^i; \dot{x}^i, \dot{x}_\alpha{}^i))$ the induced coordinate system in $TJ_p{}^1M$; if ω is given in TU by $\omega(x^i; \dot{x}^i) = (0; \omega_k{}^a(x^i)\dot{x}^k)$, or equivalently if ω writes in U as $\omega = (\omega_k{}^a dx^k)e_a$, then a routine computation, using (1.1) and (1.2), leads to the following expression for ω_1 in $J_p{}^1U$:

$$\omega_1 = (\omega_k{}^a dx^k)E_a + (x_\alpha{}^j\frac{\partial\omega_k{}^a}{\partial x^j}dx^k + \omega_k{}^a dx_\alpha{}^k)E_{a_\alpha}$$

which implies that ω_1 is a well defined $J_p{}^1V$-valued 1-form on $J_p{}^1M$. Moreover, one easily obtains the following identities, which can be considered as an alternative definition of ω_1:

$$\begin{aligned} &\omega_1(X^C)=(\omega(X))^1 \\ &\omega_1(X^{(\alpha)})=(\omega(X))^{(\alpha)}, \quad 1\leqslant\alpha\leqslant p \end{aligned} \tag{3.3}$$

for every vector field X on M.

DEFINITION 3.3. ω_1, given by (3.2) or equivalently by (3.3), will be said the ***prolongation of*** ω to $J_p{}^1M$.

This definition can be extended to higher degrees as follows: let ω be a V-valued r-form on M, $r\geqslant 2$, considered as a differentiable map $\omega: \oplus_r TM \longrightarrow TV$ which is a linear skewsymmetric map of $\oplus_r T_xM$ with values in T_0V for each $x\in M$. Define the prolongation $\omega_1: \oplus_r TJ_p{}^1M \longrightarrow TJ_p{}^1V$ of ω to $J_p{}^1M$ by setting

$$\omega_1=\alpha_V^{1,p}\cdot\omega^1\cdot\simeq\cdot(\oplus_r\alpha_M^{p,1})$$

where $\simeq: \oplus_r J_p{}^1TM \longrightarrow J_p{}^1(\oplus_r TM)$ is the canonical isomorphism. If ω is locally expressed on M by

$$\omega=(\omega^a{}_{i_1\cdots i_r}dx^{i_1}\wedge\ldots\wedge dx^{i_r})e_a$$

then ω_1 is locally expressed on $J_p{}^1M$ by

$$\begin{aligned}\omega_1=&(\omega^a{}_{i_1\cdots i_r}dx^{i_1}\wedge\ldots\wedge dx^{i_r})E_a \\ &+\left[x_\alpha{}^j\frac{\partial\omega^a{}_{i_1\cdots i_r}}{\partial x^j}dx^{i_1}\wedge\ldots\wedge dx^{i_r}\right. \\ &\left.+\sum_{k=1}^r\omega^a{}_{i_1\cdots i_r}dx^{i_1}\wedge\ldots\wedge dx_\alpha{}^{i_k}\wedge\ldots\wedge dx^{i_r}\right]E_{a_\alpha}\end{aligned}$$

Therefore, for any vector fields $X_1, \ldots, X_r$ on M,

$$\begin{aligned}&\omega_1(X_1{}^C, \ldots, X_r{}^C)=(\omega(X_1, \ldots, X_r))^1 \\ &\omega_1(X_1{}^C, \ldots, X_{j-1}{}^C, X_j{}^{(\alpha)}, X_{j+1}{}^C, \ldots, X_r{}^C)=(\omega(X, \ldots, X_r))^{(\alpha)} \\ &\omega_1(X_1{}^C, \ldots, X_i{}^{(\alpha)}, \ldots, X_j{}^{(\beta)}, \ldots, X_r{}^C)=0, \quad 1\leqslant\alpha,\beta\leqslant p\end{aligned} \tag{3.4}$$

identities which, in fact, can be considered as an alternative definition of ω_1.

The following identities are of easy proof:

$$\begin{aligned}&(\omega+\tau)_1=\omega_1+\tau_1, \quad (\lambda\omega)_1=\lambda\omega_1 \quad (\lambda\in R) \\ &i_{X^C}\omega_1=(i_X\omega)_1, \quad L_{X^C}\omega_1=(L_X\omega)_1 \\ &d\omega_1=(d\omega)_1\end{aligned} \tag{3.5}$$

Moreover, if $\phi: M \longrightarrow N$ is a differentiable map,

$$(\phi^*\omega)_1=(\phi^1)^*\omega_1 \tag{3.6}$$

for any V-valued r-form ω on N; in fact, by definition

$$(\phi^*\omega)_1=\alpha_V^{1,p}\cdot\omega^1\cdot(\oplus_r T\phi)^1\cdot\simeq\cdot(\oplus_r\alpha_M^{p,1})$$

but $(\oplus_r T\phi)^1\cdot\simeq=\simeq\cdot(\oplus_r(T\phi)^1)$ and then, using (1.3),

$$\begin{aligned}(\phi^*\omega)_1&=\alpha_V^{1,p}\cdot\omega^1\cdot\simeq\cdot(\oplus_r(T\phi)^1)\cdot(\oplus_r\alpha_M^{p,1})\\&=\alpha_V^{1,p}\cdot\omega^1\cdot\simeq\cdot(\oplus_r\alpha_N^{p,1})\cdot(\oplus_r T\phi^1)\\&=\omega_1\cdot(\oplus_r T\phi^1)=(\phi^1)^*\omega_1\end{aligned}$$

Let be $\psi: M\times G\longrightarrow M$, $\rho: G\longrightarrow Gl(V)$ and $\phi: G\times V\longrightarrow V$ as before, and for each $g\in G$, $R_g: M\longrightarrow M$ given by $R_g(x)=\psi(x,g)=xg$.

DEFINITION 3.4. A V-valued r-form $\omega:\oplus_r TM\longrightarrow TV$ is said of type (ρ, V) if, for any $g\in G$,

$$\omega\cdot(\oplus_r TR_g)=T\rho(g^{-1})\cdot\omega$$

THEOREM 3.5. *Let ω be a V-valued t-form on M of type (ρ, V). Then, the prolongation ω_1 of ω to $J_p{}^1M$ is an r-form of type $(\rho_1, J_p{}^1V)$, that is, for any $\tilde{g}\in J_p{}^1G$,*

$$\omega_1\cdot(\oplus_r TR_g)=T\rho_1(\tilde{g}^{-1})\cdot\omega_1$$

Proof. The extension for $r\geqslant 2$ being clear, we shall prove the theorem only for $r=1$. Let be $\tilde{g}\in J_p{}^1Gn$ $\omega\in T_{\tilde{x}}J_p{}^1M$, $\tilde{x}\in J_p{}^1M$, and take $\eta: R^p\longrightarrow G$ and $\varphi: R\longrightarrow J_p{}^1M$ such that $\tilde{g}=j^1\eta$, $\varphi(0)=\tilde{x}$ and $\dot{\varphi}=v$. Then, there exist a differentiable map $\psi: R\times R^p\longrightarrow M$ and a positive number δ such that $\varphi(t)=j^1\psi_t$ for $|t|<\delta$, where $\psi_t(u)=\psi^u(t)=\psi(t,u)$, $t\in R$, $u\in R^p$; therefore, for sufficiently small t, $(R_{\tilde{g}}\cdot\varphi)(t)=j^1\psi_t'$, where $\psi_t'(u)=\psi'(t,u)=\psi(t,u)\eta(u)$. Define $\Psi': R^p\longrightarrow TM$ by setting $\Psi'(u)=\dot{\varphi}'^u$; then $TR_{\tilde{g}}(v)=\widehat{(R_{\tilde{g}}\cdot\varphi)}$ and, by definition of $\alpha_M{}^{p,1}$,

$$\alpha_M^{p,1}(TR_{\tilde{g}}(v))=j^1\Psi'$$

In particular, $\alpha_M{}^{p,1}(v)=j^1\Psi$, $\Psi: R^p\longrightarrow TM$ being given by $\Psi(u)=\dot{\psi}^u$. Now,

$$\begin{aligned}(\omega\cdot\Psi')(u)&=\omega(\dot{\psi}'^u)=\omega(TR_{\eta(u)}(\dot{\psi}^u))=(\omega\text{ of type }(\rho, V))\\&=T\rho(\eta^{-1}(u))(\omega(\dot{\psi}^u))=T\rho(\eta^{-1}(u))((\omega\cdot\Psi)(u))\end{aligned}$$

Hence,

$$(\omega^1\cdot\alpha_M{}^{p,1})(TR_{\tilde{g}}(v))=\omega^1(j^1\Psi')=j^1(\omega\cdot\Psi')=j^1(T\rho(\eta^{-1})*(\omega\cdot\Psi))$$

and applying Lemma 2.3 to $f=\omega\cdot\Psi$, we deduce

$$(\alpha_V^{1,p}\cdot\omega'\cdot\alpha_M^{p,1})(TR_{\tilde{g}}(V))=\alpha_V^{1,p}(j^1(T\rho(\eta^{-1})*(\omega\cdot\Psi))=$$
$$=T\rho_1(\tilde{g}^{-1})(\alpha_V{}^{1,p}(j^1(\omega\cdot\Psi)))$$

But $j^1(\omega\cdot\Psi)=\omega^1(j^1\Psi)=\omega^1(\alpha_M^{p,1}(v))=(\omega^1\cdot\alpha_M^{p,1})(v)$, and therefore

$$\omega_1(TR_{\tilde{g}}(v))=T\rho_1(\tilde{g}^{-1})\omega_1(v))$$

All the results in this Section apply when $V=\mathfrak{g}$ is a Lie algebra. Nevertheless, the special significance of differential forms taking values in a Lie algebra, mainly in connection theory, makes suitable a slightly different approach which will be carried on in the next Section.

4. Prolongation of forms with values in a Lie algebra

Let $\mathfrak{g}$ be the Lie algebra of a Lie group G; in the sequel, we shall identify $\mathfrak{g}$ to the tangent space T_eG, e unit element of G.

DEFINITION 4.1. A $\mathfrak{g}$-valued r-form on M is a differentiable map $\omega:\oplus_r TM\longrightarrow TG$ which is a linear skew-symmetric map of $\oplus_r T_xM$ with values in T_eG for each $x\in M$.

As in Section 3, the prolongation $\omega_1:\oplus_r TJ_p{}^1M\longrightarrow TJ_p{}^1G$ of ω to $J_p{}^1M$ is defined by $\omega_1=\alpha_G^{1,p}\cdot\omega^1\cdot\simeq\cdot(\oplus_r\alpha_M{}^{p,1})$, and it is a $J_p{}^1\mathfrak{g}$-valued r-form on $J_p{}^1M$. Moreover, (3.4), (3.5) and (3.6) are still valid here, and a bracket product $[\omega,\tau]$ of $\mathfrak{g}$-valued forms ω, τ can be defined canonically. Using (3.1), one deduces

$$[\omega,\tau]_1=[\omega_1,\tau_1] \qquad (4.1)$$

Let $\rho:G\longrightarrow Aut\ (G)$ be a homomorphism of Lie groups; for each $g\in G$, $T\rho(g):TG\longrightarrow TG$ induces an automorphism $T\rho(g):\mathfrak{g}\longrightarrow\mathfrak{g}$ and, therefore, there is an induced linear representation $\rho:G\longrightarrow Aut(\mathfrak{g})$. Assume G acting on M on the right; a $\mathfrak{g}$-valued r-form ω on M will be said of type $(\rho,\mathfrak{g})$ if, for any $g\in G$,

$$\omega\cdot(\oplus_r TR_g)=T\rho(g^{-1})\cdot\omega$$

Thus, looking back to Section 3, the following natural question arises: is the prolongation ω_1 of ω of type $(\rho_1, J_p{}^1\mathfrak{g})$ for some linear representation $\rho_1:J_p{}^1G\longrightarrow Aut\ (J_p{}^1\mathfrak{g})$ canonically induced from ρ?. To answer this question we proceed as follows.

Let $Aut(J_p{}^1G)$ be the group of automorphisms of $J_p{}^1G$; then, adapting (2.2) here with the obvious changes, we define an injective homomorphism of groups

$$j_p:J_p{}^1Aut\ (G)\longrightarrow Aut\ (J_p{}^1G)$$

and thus, given $\rho : G \longrightarrow Aut(G)$ and the induced $\rho^1 : J_p{}^1G \longrightarrow J_p{}^1 Aut(G)$, we set $\rho_1 = j_p \cdot \rho^1$; now, we define the linear representation $\rho_1 : J_p{}^1G \longrightarrow Aut\ (J_p{}^1\mathfrak{g})$ by

$$\rho_1(\tilde{g}) = T\rho_1(\tilde{g}),\quad \tilde{g} \in J_p{}^1G$$

where $T\rho_1(\tilde{g}) : J_p{}^1\mathfrak{g} \longrightarrow J_p{}^1\mathfrak{g}$ is the linear map induced by $\rho_1(\tilde{g}) : J_p{}^1G \longrightarrow J_p{}^1G$. Note that there is no matter with the differentiability of j_p.

LEMMA 4.2. *Let $\eta : R^p \longrightarrow G$ be a differentiable map, $g = j^1\eta \in J_p{}^1G$, and $f : R^p \longrightarrow TG$ such that $f(R^p) \subset T_eG$. Define $T\rho(\eta^{-1}) * f : R^p \longrightarrow TG$ as in* Lemma 2.3. *Then:*

(i) $\alpha_G^{1,p}(j^1 f) \in T_{\tilde{e}} J_p{}^1G$, $\tilde{e}$ = *unit element of* $J_p{}^1G$

(ii) $\alpha_G^{1,p}(j^1(T\rho(\eta^{-1}) * f)) = T\rho_1(\tilde{g}^{-1})(\alpha_G^{1,p}(j^1 f))$

THEOREM 4.3. *Let ω be a $\mathfrak{g}$-valued r-form on M of type $(\rho, \mathfrak{g})$. Then, the prolongation ω_1 of ω to $J_p{}^1M$ is an r-form of type $(\rho_1, J_p{}^1\mathfrak{g})$.*

The proof of both Lemma 4.2 and Theorem 4.3 is similar to that of Lemma 2.3 and Theorem 3.5, respectively.

Now, assume $\rho : G \longrightarrow Aut\ (G)$ given by $\rho(g) = R_g-1 \cdot L_g$, $g \in G$; then, the induced linear representation $\rho : G \longrightarrow Aut(\mathfrak{g})$ is the adjoint representation, i.e. $T\rho(g) = ad\ g$. Then, for any $\tilde{g} = j^1\eta$, $\tilde{\mu} = j^1\mu \in J_p{}^1G$,

$$\rho_1(\tilde{g})(\tilde{\mu}) = j_p(j^1(\rho \cdot \eta))(\tilde{\mu}) = j^1((\rho \cdot \eta) * \mu)$$

where $((\rho \cdot \eta) * \mu)(t) = \rho(\eta(t))(\mu(t)) = \eta(t) \cdot \mu(t) \cdot \eta(t)^{-1} = (\eta \cdot \mu \cdot \eta^{-1})(t)$, $t \in R^p$; therefore $j^1((\rho \cdot \eta) * \mu) = \tilde{g} \cdot \tilde{\mu} \cdot \tilde{g}^{-1}$ and, hence, $\rho_1(\tilde{g}) = R_{\tilde{g}}-1 \cdot L_{\tilde{g}}$. Thus,

LEMMA 4.4. *Let $ad : G \longrightarrow Aut\ (\mathfrak{g})$ be the adjoint representation. Then $(ad)_1 : J_p{}^1G \longrightarrow Aut\ (J_p{}^1\mathfrak{g})$ is the adjoint representation of $J_p{}^1G$.*

COROLLARY 4.5. *Let ω be a $\mathfrak{g}$-valued r-form on M of type $(ad, \mathfrak{g})$. Then, the prolongation ω_1 of ω to $J_p{}^1M$ is of type $(ad, J_p{}^1\mathfrak{g})$.*

5. Prolongation of connections

Let $P(M, \pi, G)$ be a principal fibre bundle, and let $\omega : TP \longrightarrow TG$ be a $\mathfrak{g}$-valued 1-form on P. Denote by $L_x : G \longrightarrow P$ the differentiable map given by $L_x(g) = xg$, for each $x \in P$, and assume ω verifying

$$\omega(TL_x(v)) = TL_{g^{-1}}(v) \qquad (5.1)$$

for any $v \in T_gG$, $g \in G$.

LEMMA 5.1. *If* ω *verifies* (5.1), *then its prologation* ω_1 *does too, i.e.*

$$\omega_1(TLn_{\tilde{x}}(\tilde{v}))=TL_{\tilde{g}^{-1}}(\tilde{v}) \tag{5.2}$$

for every $\tilde{x}\in J_p{}^1P$, $\tilde{g}\in J_p{}^1G$ *and* $\tilde{v}\in T_{\tilde{g}}J_p{}^1G$.

Proof. Let $\varphi: R\longrightarrow J_p{}^1G$ and $\phi: R\times R^p\longrightarrow G$ be such that $\tilde{v}=\dot{\varphi}$, $\tilde{g}=\varphi(0)=j^1\phi_0$ and $\varphi(t)=j^1\phi_t$ for small t, where $\phi_t(u)=\phi^u(t)=\phi(t, u)$, $t\in R$, $u\in R^p$. Define $\phi': R\times R^p\longrightarrow G$ by $\phi'(t, u)=\phi(0, u)^{-1}\phi(t, u)$; then $(L_{\tilde{g}^{-1}}\cdot\varphi)(t)=j^1\phi_t'$ and, if we set $\Psi': R^p\longrightarrow TG$ given by $\Psi'(u)=\dot{\phi}'^u$,

$$\alpha_G{}^{p,1}(TL_{\tilde{g}^{-1}}(\tilde{v}))=j^1\Psi' \tag{5.3}$$

On the other hand, let $\eta: R^p\longrightarrow P$ be such that $\tilde{x}=j^1\eta$, and define $\eta': R\times R^p\longrightarrow P$ by $\eta'(t, u)=\eta(u)\phi(t, u)$; then $(L_{\tilde{x}}\cdot\varphi)(t)=j^1\eta_t'$ for small t, and if we set $Y: R^p\longrightarrow TP$ given by $Y(u)==\dot{\eta}'^u$, then $\alpha_P{}^{p,1}(TL_{\tilde{x}}(\tilde{v}))=j^1Y$. Therefore,

$$(\omega^1\cdot\alpha_P{}^{p,1})(TL_{\tilde{x}}(\tilde{v}))=j^1(\omega\cdot Y) \tag{5.4}$$

Now, since $\eta'^u(t)=\eta(u)\phi^u(t)$, we have $\dot{\eta}'^u=TL_{\eta(u)}(\dot{\phi}^u)$, and hence

$$(\omega\cdot Y)(u)=\omega(TL_{\eta(u)}(\dot{\phi}^u))=(\text{by } (5.1))=TL_{\phi^u(0)^{-1}}(\dot{\phi}^u)$$

On the other hand $\phi'^u(t)=\phi^u(0)^{-1}\phi^u(t)$; therefore $\Psi'(u)=(\omega\cdot Y)(u)$ and hence $\Psi'=\omega\cdot Y$. Finally, by (5.3) and (5.4), $(\omega^1\cdot\alpha_P{}^{p,1})(TL_{\tilde{x}}(\tilde{v}))=\alpha_G{}^{p,1}(TL_{\tilde{g}^{-1}}(\tilde{v}))$ and (5.2) is proved.

Let Γ be a connection on P whose connection form will be denoted ω; following [4], ω can be seen as a $\mathfrak{g}$-valued 1-form on P of type $(ad, \mathfrak{g})$ which satisfies (5.1). Let ω_1 be the prolongation of ω to $J_p{}^1P$; from Corollary 4.5 and Lemma 5.1, we deduce

THEOREM 5.2. ω_1 *defines a connection* Γ_1 *on* $J_p{}^1P(J_p{}^1M, \pi^1, J_p{}^1G)$, *which will be said the prolongation of connection* Γ *on* P.

This Theorem has been proved also in [3] through a slightly different procedure.

THEOREM 5.3. *Let* Ω *be the curvature form of connection* Γ *on* P. *Then, the prolongation* Ω_1 *of* Ω *is the curvature form of the prolongation* Γ_1 *of* Γ. *Therefore,* Γ *is flat if and only if* Γ_1 *is flat.*

Proof. By virtue of (3.5) and (4.1), and the structure equations, we have

$$\Omega_1=(d\omega)_1+(1/2)[\omega, \omega]_1=d\omega_1+(1/2)[\omega_1, \omega_1].$$

Next, with a view on further applications, we shall search for the relation between the operators of exterior covariant differentiation D, D_1 with respect to Γ and Γ_1, respectively. So, let V be again a vector space, $\rho : G \longrightarrow Gl(V)$ a linear representation and τ a V-valued r-form on P of type (ρ, V); recall that τ is said *tensorial* if, moreover, $\tau(X_1, \ldots, X_r)=0$ whenever at least one of the tangent vectors X_i of P is vertical.

PROPOSITION 5.4. *The prolongation τ_1 of a tensorial form τ on P of type (ρ, V) is also a tensorial form on $J_p{}^1P$.*

Proof. Routine, taking into account the local expressions of τ and τ_1.

Let X^H denote the horizontal lift of the vector field X with respect to Γ or Γ_1 indistinctly.

PROPOSITION 5.5. *For any vector field X on M,*

$$(X^C)^H=(X^H)^C, \quad (X^{(\alpha)})^H=(X^H)^{(\alpha)}, \quad 1\leqslant\alpha\leqslant p \tag{5.5}$$

Proof. From (3.3) we have $\omega_1((X^H)^C)=(\omega(X^H))^1=0$, $\omega_1((X^H)^{(\alpha)})=(\omega(X^H))^{(\alpha)}=0$ and therefore $(X^H)^C$, $(X^H)^{(\alpha)}$ are all horizontal with respect to Γ_1. Hence, it suffices to check that, at any point $\tilde{x}\in J_p{}^1P$,

$$T\pi^1((X^H)_{\tilde{x}}{}^C)=(X^C)_{\pi^1(\tilde{x})}, \quad T\pi^1((X^H)_{\tilde{x}}{}^{(\alpha)})=(X^{(\alpha)})_{\pi^1(\tilde{x})}$$

which can be done using local expressions.

PROPOSITION 5.6. *For any V-valued r-form τ on P of type (ρ, V)*

$$D_1\tau_1=(D\tau)_1 \tag{5.6}$$

Proof. Since $(D\tau)_1$ is tensorial, it suffices to check the identity applying both members to horizontal arguments of the form $(X^C)^H$, $(X^C)^{(\alpha)}$ for arbitrary vector field X on M; the result follows directly from the definition of D and D_1 taking into account (3.4), (3.5) and (5.5).

Note that Theorem 5.3 is also a consequence of this Proposition.

Next, assume that Γ is a linear connection on M, i.e. a connection on $FM(M, \pi_M, Gl(n))$; then, Γ_1 is a connection on $J_n{}^1FM(J_n{}^1M, \pi^1, J_n{}^1Gl(n))$, and it induces a linear connection $\tilde{\Gamma}$ on FM as follows: let $j_M : J_n{}^1FM \longrightarrow FJ_n{}^1M$ be the canonical embedding, and let $\tilde{\Gamma}$ be the unique linear connection on $J_n{}^1M$ whose connection form $\tilde{\omega}$ verifies

$$j_M{}^*\tilde{\omega}=Tj_n\cdot\omega_1$$

Then, the linear connection on FM is obtained by restricting $\tilde{\Gamma}$ to the open submanifold $FM \subset J_n{}^1M$, and it will be still denoted by $\tilde{\Gamma}$ and its connection form by $\tilde{\omega}$. Since we have proved in [3] that the covariant derivation defined on FM by $\tilde{\Gamma}$ coincides with the complete lift to FM [6] of the covariant derivation defined by Γ on M, $\tilde{\Gamma}$ will be said the *complete lift* of Γ to FM.

Let $\tilde{\Omega}, \Omega$ be the curvature forms of $\tilde{\omega}, \omega$ respectively; then $j_M{}^*\tilde{\Omega} = Tj_n \cdot \Omega$ and hence

COROLLARY 5.7. *The complete lift $\tilde{\Gamma}$ of Γ to FM is flat if and only if Γ is flat.*

Let θ_M, $\theta_{J_n{}^1M}$ be the canonical 1-forms of FM and $FJ_n{}^1M$, respectively.

LEMMA 5.8. $j_M^*\theta_{J_n{}^1M} = (\theta_M)_1$

Proof. Straightforward computation using local expressions.

Let $\Theta = D\theta_M$ be the torsion form of Γ; then

$$\Theta_1 = (D\theta_M)_1 = D_1(\theta_M)_1 = D_1(j_M^*\theta_{J_n{}^1M}) = j_M^*(\tilde{D}\theta_{J_n{}^1M}) = j_M^*\tilde{\Theta}$$

where $\tilde{D}$ denotes the exterior covariant differentiation with respect to $\tilde{\Gamma}$ and $\tilde{\Theta}$ is the torsion form of $\tilde{\omega}$. Restricting once more to FM, we get

THEOREM 5.9. *Let Θ, $\tilde{\Theta}$ be the torsion forms of $\omega, \tilde{\omega}$ respectively. Then $j_M^*\tilde{\Theta} = \Theta_1$ and hence Γ is torsionfree if and only if $\tilde{\Gamma}$ is torsionfree too.*

Let us remark that Mok [6] has proved similar results for the torsion and curvature tensors of Γ and $\tilde{\Gamma}$.

6. Complete lift of tensor fields of types $(0, s)$ and $(1, s)$

Let V, $\tilde{V}$ be vector spaces, $\rho : Gl(n) \longrightarrow Gl(V)$, $\tilde{\rho} : Gl(n+n^2) \longrightarrow Gl(\tilde{V})$ linear representations and $i : J_n{}^1V \longrightarrow \tilde{V}$ a homomorphism such that

$$i \cdot \rho_1(\tilde{g}) = \tilde{\rho}(j_n(\tilde{g})) \cdot i \tag{6.1}$$

for every $\tilde{g} \in J_n{}^1Gl(n)$.

Let $t : FM \longrightarrow V$ be a differentiable function of type (ρ, V); from Theorem 3.2 we know that the prolongation t^1 of t to $J_n{}^1FM$ is a differentiable function of type (ρ_1, J_n^1V). Then, given $\tilde{y}\in FJ_n^1M$, choose $\tilde{x}\in J_n^1FM$ and $\tilde{a}\in Gl(n+n^2)$ such that $\tilde{y}=j_M(\tilde{x})\tilde{a}$, and define $\tilde{t}: FJ_n^1M\longrightarrow\tilde{V}$ by setting

$$\tilde{t}(\tilde{y})=\tilde{\rho}(\tilde{a}^{-1})((i\cdot t^1)(\tilde{x})) \tag{6.2}$$

Identity (6, 1) above implies that $\tilde{t}(\tilde{y})$ does not depend on the choice of $\tilde{x}$ and $\tilde{a}$; moreover, $\tilde{t}$ is obviously of type $(\tilde{\rho}, \tilde{V})$ and, then, we can adopt the following definition:

DEFINITION 6.1 The differentiable function $\tilde{t}: FJ_n^1M\longrightarrow\tilde{V}$ given by (6.1) will be called the *complete lift* of t to FJ_n^1M.

Let Γ be a linear connection on M, Γ_1 and $\tilde{\Gamma}$ the induced connections on J_n^1FM and FJ_n^1M, respectively, and D, D_1, $\tilde{D}$ their respective exterior covariant differentials.

PROPOSITION 6.2. *Let $\tilde{t}$ be the complete lift of a differentiable function $t: FM\longrightarrow V$ of type (ρ, V). Then.*

$$j_M^*(\tilde{D}\tilde{t})=i\cdot(Dt)_1$$

Proof. (6.2) implies $\tilde{t}\cdot j_M=i\cdot t^1$ and hence $j_M{}^*(d\tilde{t})=i\cdot(dt^1)$; then, since j_M preserves horizontal vectors, $j_M{}^*(\tilde{D}\tilde{t})=i\cdot(Dt^1)$, and the result follows from (5.6).

COROLLARY 6.3. *$Dt\equiv0$ implies $\tilde{D}\tilde{t}\equiv0$.*

Proof. Obviously $Dt\equiv0$ implies $j_M{}^*(\tilde{D}\tilde{t})\equiv0$; since $\tilde{D}\tilde{t}$ is of type $(\tilde{\rho}, \tilde{V})$, one follows $\tilde{D}\tilde{t}\equiv0$.

Remark that the converse of this Corollary is not true in general.

This procedure above of "lifting" differentiable functions on FM to FJ_n^1M has some interesting applications when particular choices of V, $\tilde{V}$, ρ, $\tilde{\rho}$ and i are done; in fact, it provides a geometrical interpretation of the (so called) complete lift of tensor fields on M to FM introduced in [1] and [6], as well as of the prolongation to FM of G-structures on M defined by tensor fields which has been studied in [2].

In order to give that interpretation, let us firstly recall the one-to-one correspondence between tensor fields H on M of type (r, s) and differentiable functions $t: FM\longrightarrow\otimes_s{}^rR^n$ of type $(\rho, \otimes_s{}^rR^n)$, ρ being the canonical linear representation of $Gl(n)$ into $\otimes_s{}^rR^n$; let (U, x^i) be a coordinate system in M, $(FU, x^i, X_j{}^i)$ the induced one in FM, and assume H (resp. t) given in U (resp. in FU) by its components

$H^{j_1\cdots j_r}_{i_1\cdots i_s}$ (resp. $t^{j_1\cdots j_r}_{i_1\cdots i_s}$); then,

$$t^{k_1\cdots k_r}_{h_1\cdots h_s}(x^i,\ X_j{}^i)=H^{j_1\cdots j_r}_{i_1\cdots i_s}(x^i)X_{h_1}{}^{i_1}\cdots X_{h_s}{}^{i_s}\overline{X}_{j_1}{}^{k_1}\cdots\overline{X}_{j_r}{}^{k_r} \tag{6.3}$$

where $(\overline{X}_j{}^i)=(X_j{}^i)^{-1}$.

(I) *Complete lift of functions*

First of all, note that given a differentiable function $f: M \longrightarrow R$, then $t=f\cdot\pi_M : FM \longrightarrow R$ is of type (ρ, R), $\rho : Gl(n) \longrightarrow Gl(R)$ being he trivial representation; conversely, if $t: FM \longrightarrow R$ is of type (ρ, R), then t projects down to a differentiable function $f: M \longrightarrow R$.

So, take $V=\tilde{V}=R$, ρ and $\tilde{\rho}$ being the trivial representations, and $i=\pi$ given by (2.4); then, (6.1) holds trivially, and for any $f: M \longrightarrow R$ we can consider the complete lift $\tilde{t}: FJ_n{}^1M \longrightarrow R$ of $t=f\cdot\pi_M$. Thus, given $\tilde{y}\in FJ_n{}^1M$, choose $\tilde{x}=(x^i,\ X_j{}^i,\ x_\alpha{}^i,\ X_{j\alpha}{}^i)\in J_n{}^1FM$ and $\tilde{a}\in Gl(n+n^2)$ such that $\tilde{y}=j_M(\tilde{x})\tilde{a}$; an straightforward computation leads to

$$\tilde{t}(\tilde{y})=\sum_{\alpha=1}^{n} x_\alpha{}^j \frac{\partial f}{\partial x^j}$$

and, then, the projection down to $J_n{}^1M$ of $\tilde{t}$ is the differentiable function $\tilde{f}: J_n{}^1M \longrightarrow R$ given by

$$\tilde{f}(x^i, x_\alpha{}^i)=\sum_{\alpha=1}^{n} x_\alpha{}^j \frac{\partial f}{\partial x^j}$$

function which will be said the complete lift of f to $J_n{}^1M$. The restriction $f^C=\tilde{f}_{|FM}$ is just the so called complete lift of f to FM in [1].

(II) *Complete lift of tensor fields of type* $(1, s)$

Let be $V=\otimes_s{}^1R^n$, $\tilde{V}=\otimes_s{}^1(J_n{}^1R^n)\simeq\otimes_s{}^1R^{n+n^2}$, ρ and $\tilde{\rho}$ the canonical linear representations of $Gl(n)$ and $Gl(n+n^2)$ into V and $\tilde{V}$, respectively, and i being the homomorphism given in (I), Section 2, for $p=n$. From (2.7) we know that (6.1) holds and hence, if H is a tensor field on M of type $(1, s)$ with associate function $t: FM \longrightarrow \otimes_s{}^1R^n$ of type $(\rho, \otimes_s{}^1R^n)$, $\tilde{t}: FJ_n{}^1M \longrightarrow \otimes_s{}^1R^{n+n^2}$ is well defined by (6.2) and determines a tensor field $\tilde{H}$ on $J_n{}^1M$ of type $(1, s)$.

In order to compute $\tilde{H}$, assume $\tilde{y}=j_M(\tilde{x})$ with $\tilde{x}=(x^i, X_j{}^i, x_\alpha{}^i, X_{j\alpha}{}^i)\in J_n{}^1FM$; then, taking into account (1.5), (2.5), (6.2) and (6.3), we obtain

$$\tilde{t}(\tilde{y})=H^{i}{}_{j_1\cdots j_s}X_{k_1}{}^{j1}\cdots X_{k_s}{}^{js}\bar{X}_i{}^{h}E_h\otimes E^{k1}\otimes\cdots\otimes E^{k_s}$$
$$+[x_\alpha{}^{k}X_{k_1}{}^{j1}\cdots X_{k_s}{}^{js}\bar{X}_i{}^{h}\partial_k H_{j_1}{}^{i}{}_{\cdots j_s}-X_{1\alpha}{}^{k}X_{k_1}{}^{j1}\cdots X_{k_s}{}^{js}\bar{X}_k{}^{h}\bar{X}_i{}^{1}H_{j_1}{}^{i}{}_{\cdots j_s}$$
$$+\sum_{1=1}^{s}X_{k1\alpha}{}^{j1}X_{k_1}{}^{j1}\cdots X_{k_{1-1}}{}^{j_{l}-1}X_{k_{1+1}}{}^{j_l+1}\cdots X_{k_s}{}^{js}\bar{X}_i{}^{h}H_{j_1}{}^{i}{}_{\cdots j_s}]$$
$$E_{h_\alpha}\otimes E^{k1}\otimes\ldots\otimes E^{k_s}$$
$$+\sum_{1=1}^{s}\delta^{\alpha\beta}X_{k_1}{}^{j1}\cdots X_{k_s}{}^{js}\bar{X}_i{}^{h}H_{j_1}{}^{i}{}_{\cdots j_s}E_{h_\alpha}\otimes E^{k1}\otimes\cdots\otimes E_\beta{}^{k1}\otimes\cdots\otimes E^{k_s}$$

where $H_{j_1}{}^{i}{}_{\cdots j_s}$ are the local components of H and $\partial_k=(\partial/\partial x^k)$. Then, using again (6.3) with the appropriate changes to $J_n{}^1M$ and $FJ_n{}^1M$, one obtains

$$\tilde{H}(x^i,\ \ x_\alpha{}^i)=H_{j_1}{}^{i}{}_{\cdots j_s}\frac{\partial}{\partial x^i}\otimes dx^{j1}\otimes\cdots\otimes dx^{js}$$
$$+(x_\alpha{}^{h}\partial_h H_{j_1}{}^{i}{}_{\cdots j_s})\frac{\partial}{\partial x_\alpha{}^i}\otimes dx^{j1}\otimes\cdots\otimes dx^{js}$$
$$+\sum_{1=1}^{s}\delta^{\alpha\beta}H_{j_1}{}^{i}{}_{\cdots j_s}\frac{\partial}{\partial x_\alpha{}^i}\otimes dx^{j1}\otimes\cdots\otimes dx_\beta{}^{j1}\otimes\ldots\otimes dx^{js}$$

Note that if X is a vector field on M, then $\tilde{X}$ is just the complete lift X^C of X to $J_n{}^1M$; moreover, the restriction $H^C=\tilde{H}_{|FM}$ is nothing but the complete lift of H to FM which has been defined in [6].

Also, remark that the converse of Corollary 6.3 holds in this case because i is injective. Hence, we can state

COROLLARY 6.4. *Let Γ be a linear connection and H a tensor field of type $(1, s)$ on M, and $\tilde{\Gamma}$, H^C their complete lifts to FM. Then, H is parallel with respect to Γ if and only if H^C is parallel with respect to $\tilde{\Gamma}$.*

See Mok [6] for a different proof of this Corollary.

(III) *Complete lift of tensor fields of type* $(0, s)$

Let be $V=\otimes_s R^n$, $\tilde{V}=\otimes_s(J_n{}^1R^n)\simeq\otimes_s R^{n+n^2}$, ρ and $\tilde{\rho}$ the canonical linear representations of $Gl(n)$ and $Gl(n+n^2)$ into V and $\tilde{V}$, respectively, and take i being given as in (II), Section 2, for $p=n$. Once more (6.1) holds, and if H is a tensor field on M of type $(0, s)$ with associate function $t: FM\longrightarrow\otimes_s R^n$, then the complete lift $\tilde{t}$, given by (6.2), writes locally as follows: assume $\tilde{y}, \tilde{x}$ as before, then

$$\tilde{t}(\tilde{y})=\sum_{\alpha=1}^{n}\{(x_\alpha{}^{h}X_{k_1}{}^{i1}\cdots X_{k_s}{}^{is}\partial_h H_{i_1\cdots i_s}$$
$$+\sum_{1=1}^{s}X_{k_1}{}^{i1}{}_{\alpha}X_{k_1}{}^{i1}\cdots X_{k_{1-1}}{}^{i_l-1}X_{k_{1+1}}{}^{i_l+1}\cdots X_{k_s}{}^{is})E^{k1}\otimes\cdots\otimes E^{k_s}$$

$$+\sum_{l=1}^{s} X_{k_1}{}^{i_1}\cdots X_{k_s}{}^{i_s} H_{i_1\cdots i_s} E^{k_1}\otimes\cdots\otimes E_\alpha{}^{k_l}\otimes\cdots\otimes E^{k_s}\}$$

where $H_{i_1\cdots i_s}$ are the local components of H. Hence, the local expression for $\tilde{H}$ is:

$$\tilde{H}(x^i,\ x_\alpha{}^i)=\sum_{\alpha=1}^{n}\{x_\alpha{}^h\partial_h H_{i_1\cdots i_s} dx^{i_1}\otimes\cdots\otimes dx^{i_s}$$

$$+\sum_{l=1}^{s} H_{i_1\cdots i_s} dx^{i_1}\otimes\cdots\otimes dx_\alpha{}^{i_l}\otimes\cdots\otimes dx^{i_s}\}$$

Note that the restriction $H^C=\tilde{H}_{|FM}$ is just the complete lift of H to FM which has been defined in [1].

Also, remark that the converse of Corollary 6. 3 still holds in this case, because $\tilde{D}\tilde{t}\equiv 0$ implies that $i(Dt)_1(X^C)^H=0$ and $i(Dt)_1(X^{(\alpha)})^H=0$ for any vector field X on M, and then, by virtue of (3. 3) and (5. 5), $i((Dt)X^H)^1=0$ and $i((Dt)X^H)^{(\alpha)}=0$ or equivalently $i(X^H t)^1=0$ and $i(X^H t)^{(\alpha)}=0$; finelly (2. 6) implies $X^H t=0$ and hence $Dt=0$. Therefore, Corollary 6. 4 can be enlarged including the type $(0, s)$.

7. Prolongation to *FM* of *G*-structures defined by tensor fields

Let $u\in V$ be a fixed element of V, G_u the isotropy group of u with respect to the linear representation $\rho: Gl(n)\longrightarrow Gl(V)$, and $V_u=\{\rho(g)u/g\in Gl(n)\}$. There exists an one-to-one correspondence between G_u-structures P_{G_u} on M and differentiable functions $t: FM\longrightarrow V$ of type (ρ, V) such that: (i) $t(FM)\subset V_u$; (ii) t is a differentiable map of FM into V_u; this correspondence is given by setting $P_{G_u}=t^{-1}(u)$.

Let $\tilde{\rho}: Gl(n+n^2)\longrightarrow Gl(\tilde{V})$ be another linear representation, and let $i: J_n{}^1 V\longrightarrow\tilde{V}$ be a homomorphism such that (6. 1) holds. Let $u^1\in J_n{}^1 V$ be the 1-jet at 0 of the constant map of R^n into $u\in V$, and denote $\tilde{u}=i(u^1)$; then, from (2. 2), (2. 3) and (6. 1), we deduce

$$\tilde{G}_u=j_n(J_n{}^1 G_u)\subset G_{\tilde{u}} \tag{7. 1}$$

$G_{\tilde{u}}$ being the isotropy group of $\tilde{u}$.

Let $t: FM\longrightarrow V$ define the G_u-structure P_{G_u} on M and $\tilde{t}: FJ_n{}^1 M\longrightarrow\tilde{V}$ the complete lift of $\tilde{t}$.

LEMMA 7. 1. $\tilde{t}(FJ_n{}^1 M)\subset\tilde{V}_{\tilde{u}}$

Proof. Firstly, note that $t^1(\tilde{x})\in J_n{}^1 V_u$ for any $\tilde{x}\in J_n{}^1 FM$. Next, given an arbitrary $j^1 h\in J_n{}^1 V_u$, define $\tilde{g}\in J_n{}^1\ Gl(n)$ as $\tilde{g}=j^1\eta$, where

$\eta : R^n \longrightarrow Gl(n)$ is determined by the identity $h(t) = \rho(\eta(t))u$, $t \in R^n$. Then, from (2. 2) and (2. 3), one follows $j^1h = \rho_1(\tilde{g})(u^1)$; hence $i(j^1h) = \tilde{\rho}(j_n(\tilde{g}))(\tilde{u})$, and the result follows from (6. 2).

Thus, $P_{G_{\tilde{u}}} = \tilde{t}^{-1}(\tilde{u})$ defines a $G_{\tilde{u}}$-structure on $J_n{}^1M$ and hence, by restriction, on FM; so, we can state

THEOREM 7. 2. *If M admits a G_u-structure defined by a tensor t of type (ρ, V) and V_u-valued, then the complete lift $\tilde{t}$ of t defines a $G_{\tilde{u}}$-structure on FM.*

This Theorem improves some particular results stated in [2]. Let $\tilde{P}_{G_u} = (j_M \cdot j^1)(J_n{}^1P_{G_u})_{|FM}$ be the prolongation of P_{G_u} to FM; obviously $\tilde{P}_{G_u} \subset P_{G_{\tilde{u}}|FM}$, and thus $P_{G_{\tilde{u}}|FM}$ is isomorphic to the canonical prolongation of $\tilde{P}_{G_u}$ induced by the injection $\tilde{G}_u \longrightarrow G_{\tilde{u}}$.

(I) *G-structures defined by tensor fields of types* (1, s) *and* (0, s)

Assume $V, \tilde{V}, \rho, \tilde{\rho}$ and i being the same as in (II) or in (III), Section 6. Then, for a fixed $u \in V$, let $t : FM \longrightarrow V$ of type (ρ, V) define a G_u-structure P_{G_u} on M, and denote by H the associate tensor field on M of type $(1, s)$ or $(0, s)$, tensor field which will be said to define P_{G_u}. Then, combining Theorem 7. 2 with the results in Section 6, we can assert that the complete lift H^C of H to FM defines $P_{G_{\tilde{u}}|FM}$, and conversely. This result has been proved in [2] (Theorems 5. 4 and 5. 10) for tensor fields of type (1, 1) and (0, 2).

(II) *Prolongation of volume forms*

Let be $V = R$ and $\rho : Gl(n) \longrightarrow Gl(R)$ given by $\rho(g)\xi = (det\ g)\xi$, $g \in Gl(n)$, $\xi \in R$. Then, if $u = 1 \in R$, $G_u = Sl(n)$, the real special linear group. To give an $Sl\ (n)$-structure P on M is equivalent to give a differentiable function $t : FM \longrightarrow V$ of type (ρ, V) or to give a volume form Ω on M, the relation between Ω and t being given by

$$\Omega_x(X_1, \ \ldots, \ X_n) = \lambda(t(y))(y^{-1}X_1, \ \ldots, \ y^{-1}X_n)$$

for any $X_1, \ldots, X_n \in T_xM$, $x \in M$, $y \in FM$ such that $\pi_M(y) = x$, and $\lambda : R \longrightarrow \bigwedge^n(R^n)^*$ being the canonical isomorphism given by $\lambda\ (\xi) = \xi e^1 \wedge \ldots \wedge e^n$, $\xi \in R$.

Now, consider $J_n{}^1R \simeq R^{1+n}$, $u^1 \in J_n{}^1R$ the 1-jet at 0 of the constant map onto $u = 1$, $\tilde{V} = R$, $\tilde{\rho} : Gl(n+n^2) \longrightarrow Gl(R)$ also defined by the determinant and $i = \pi_R$ the target map. Then (2. 7) is still verified, and, for $\tilde{u}=i(u^1)$, $G_{\tilde{u}}=Sl(n+n^2)$. Therefore, from Theorem 7.2 we deduce that, given a volume form Ω on M, there exists canonically associated a volume form $\tilde{\Omega}$ on FM.

In order to compute $\tilde{\Omega}$ from Ω, we proceed as follows: let (U, x^i) be a coordinate system in M and assume Ω given in U by $\Omega=fdx^1\wedge\ldots\wedge dx^n$, f being nowhere zero on U; then the associate function t is given by $t=f\cdot\pi_M$. If $\tilde{t}$ denotes the complete lift of t,

$$\tilde{\Omega}_{\tilde{x}}\left(\ldots, \frac{\partial}{\partial x^i}, \ldots, \frac{\partial}{\partial x_\alpha{}^i}, \ldots\right)$$
$$=\lambda(\tilde{t}(\tilde{y}))\left(\ldots, \tilde{y}^{-1}\left(\frac{\partial}{\partial x^i}\right), \ldots, \tilde{y}^{-1}\left(\frac{\partial}{\partial x_\alpha{}^i}\right), \ldots\right)$$

for any $\tilde{x}\in J_n{}^1 U$ and $\tilde{y}\in FJ_n{}^1U$ such that $\pi_{J^1{}_nM}(\tilde{y})=\tilde{x}$; and, if we take $\tilde{y}=j_M(x^i, X_j{}^i, x_\alpha{}^i, X_{j\alpha}{}^i)$ with $(x^i, X_j{}^i, x_\alpha{}^i, X_{j\alpha}{}^i)\in J_n{}^1P_{\mid J_n{}^1U}$, P being the $Sl(n)$-structure determined by Ω, then $det(X_j{}^i)=1$ and $\tilde{t}(\tilde{y})=f(x^i)$. Therefore,

$$\tilde{\Omega}_{\tilde{x}}\left(\ldots, \frac{\partial}{\partial x^i}, \ldots, \frac{\partial}{\partial x_\alpha{}^i}, \ldots\right)$$
$$=f(x^i)e^1\wedge\ldots\wedge e^{n+n^2}\left(\ldots, \bar{X}_i{}^je_j-\sum_{\beta=1}^{n}\bar{X}_i{}^hX_{h\beta}{}^k\bar{X}_k{}^je_{\beta n+j}, \ldots, \sum_{\beta=1}^{n}\delta_\alpha{}^\beta\bar{X}_i{}^je_{\beta n+j}, \ldots\right)$$
$$=f(x^i)\,(det(\bar{X}_j{}^i))^{n+1}=f(x^i)$$

Thus, $\tilde{\Omega}$ is locally given in $J_n{}^1U$ by

$$\tilde{\Omega}=fdx^1\wedge\ldots\wedge dx^n\wedge dx_1{}^1\wedge\ldots\wedge dx_n{}^n$$

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