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A NOTE ON A ONE-PARAMETER ADDITIVE FAMILY OF OPERATORS DEFINED ON ANALYTIC FUNCTIONS

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1. Introduction

Let $\mathcal{J}$ denote the class of functions of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n \tag{1.1}$$

which are analytic in the unit disk $\mathcal{U}=\{z:|z|<1\}$ and σ be a probability measure supported by the closed interval $[0,1]$. Further let L denote the linear integral transformation

$$Lf(z)=\int_0^1 \frac{f(zt)}{t}d\sigma(t). \tag{1.2}$$

Since it is evident that $f(z)\in\mathcal{J}$ implies $Lf(z)\in\mathcal{J}$, we can apply the operator L successively for obtaining

$$L^n f(z)=LL^{(n-1)}f(z) \qquad (n\in\mathcal{N}=\{1,2,3\ldots\}) \tag{1.3}$$

with $L^0 f(z)=f(z)$.

Recently Komatu [2] interpolated the sequence $\{L^n\}$ into a family $\{L^\lambda\}$ depending on a continuous parameter $\lambda\geqq 0$ such that the condition of additivity

$$L^\lambda L^\mu=L^{\lambda+\mu} \tag{1.4}$$

is satisfied and showed the series expansion of $L^\lambda f(z)$ applied to $f(z)\in\mathcal{J}$ is obtained in the form

$$L^\lambda f(z)=z+\sum_{n=2}^{\infty}\frac{a_n}{n^\lambda}z^n \tag{1.5}$$

in case of $\sigma(t)=t$.

Let $\mathcal{S}$ denote the class of functions defined by (1.1) which are analytic and univalent in the unit disk $\mathcal{U}$. A function $f(z)\in\mathcal{S}$ is said to be starlike with respect to the origin in the unit disk $\mathcal{U}$ if, and only if,

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(1.6) $$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>0 \qquad (z\in\mathcal{U}).$$

The concept of order for a starlike function has been introduced by Holland and Thomas [1], Libera [3], Padmanabhan [6] and Robertson [7]. According to Padmanabhan [6], a function $f(z)\in\mathcal{S}$ is said to be starlike of order α $(0<\alpha\leqq 1)$ in the unit disk $\mathcal{U}$ if

(1.7) $$\left|\left(\frac{zf'(z)}{f(z)}-1\right)\Big/\left(\frac{zf'(z)}{f(z)}+1\right)\right|<\alpha \qquad (z\in\mathcal{U})$$

for some α $(0<\alpha\leqq 1)$. We denote the class of such functions by $\mathcal{S}(\alpha)$. For the class $\mathcal{S}(\alpha)$, Padmanabhan [6] has showed representation formula, distortion theorems and the radius of convexity, Mogra [4] has obtained a coefficient theorem and a sufficient condition and Owa [5] have showed some distortion theorems for $F_p(z)$ defined by Hadamard product of functions in $\mathcal{S}(\alpha)$.

2. Komatu's conjectures

Let $\mathcal{S}^*$ and $\mathcal{K}$ denote the classes of functions $f(z)$ defined by (1.1) analytic and starlike with respect to the origin in the unit disk $\mathcal{U}$ and analytic and convex in the unit disk $\mathcal{U}$, respectively.

Recently Komatu [2] gave the following conjectures for these classes.

CONJECTURE 1. *If $f(z)$ is in the class $\mathcal{S}$, then $L^\lambda f(z)\in\mathcal{S}$ at least for $\lambda\geqq 1$.*

CONJECTURE 2. *If $f(z)$ is in the class $\mathcal{K}$ (or, more generally, $f(z)\in\mathcal{S}^*$), then $L^\lambda f(z)\in\mathcal{K}$ at least for $\lambda\geqq 1$.*

In this section, we consider the Komatu's conjectures for the class $\mathcal{S}(\alpha)$. We need the following lemmas by Mogra [4].

LEMMA 1. *If the function $f(z)$ defined by* (1.1) *is in the class $\mathcal{S}(\alpha)$ for some α $(0<\alpha\leqq 1)$, then for $\alpha=1$*

(2.1) $$|a_n|\leqq n \qquad (n\geqq 2)$$

while, for $0<\alpha<1$,

(2.2) $$|a_n|\leqq n\alpha^{n-1} \qquad (n=2,3,4,\dots,N)$$

and

(2.3) $$|a_n|\leqq\frac{1}{(n-1)}\alpha^N N(N+1) \qquad (n>N),$$

where

$$N=\left[\frac{1+\alpha}{1-\alpha}\right] \tag{2.4}$$

and [ ] *means the Gauss' symbol. The estimates in* (2.1) *and* (2.2) *are sharp.*

LEMMA 2. *Let the function* $f(z)$ *defined by* (1.1) *be analytic in the unit disk* $\mathcal{U}$. *If, for some* α $(0<\alpha\leqq 1)$,

$$\sum_{n=2}^{\infty}\left\{\left(\frac{1+\alpha}{2\alpha}\right)n+\frac{\alpha-1}{2\alpha}\right\}|a_n|\leqq 1, \tag{2.5}$$

then $f(z)$ *belongs to the class* $\mathcal{S}(\alpha)$.

THEOREM 1. *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $\mathcal{S}(1/3)$. *Then* $L^{\lambda}f(z)$ *belongs to the same class* $\mathcal{S}(1/3)$ *at least for* $\lambda\geqq\lambda_0$ *where* λ_0 *is a certain number less than* 23/8.

Proof. Putting $\alpha=1/3$ in (2.5) and using (1.5), we can see that

$$\begin{aligned}\sum_{n=2}^{\infty}(2n-1)\frac{|a_n|}{n^{\lambda}}&=\frac{3|a_2|}{2^{\lambda}}+\sum_{n=3}^{\infty}(2n-1)\frac{|a_n|}{n^{\lambda}}\\ &\leqq\frac{1}{2^{\lambda-1}}+\frac{2}{3}\sum_{n=3}^{\infty}\frac{2n-1}{n^{\lambda}(n-1)}\\ &<\frac{1}{2^{\lambda-1}}+\frac{2}{3}\sum_{n=3}^{\infty}\frac{1}{n^{\lambda-1}}\\ &<\frac{1}{2^{\lambda-1}}+\frac{2}{3}\left\{\frac{2^{\lambda-2}}{2^{\lambda-2}-1}-\left(1+\frac{1}{2^{\lambda-1}}\right)\right\}\\ &=\frac{1}{3\cdot 2^{\lambda-1}}+\frac{2}{3(2^{\lambda-2}-1)}\\ &<1\end{aligned} \tag{2.6}$$

for any real $\lambda\geqq 23/8$ with the aid of Lemma 1, because

$$1+\sum_{n=2}^{\infty}\frac{1}{n^{\lambda-1}}<\frac{2^{\lambda-2}}{2^{\lambda-2}-1} \tag{2.7}$$

for $\lambda>2$. This shows that $L^{\lambda}f(z)\in\mathcal{S}(1/3)$ by means of Lemma 2.

THEOREM 2. *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $\mathcal{S}(1/2)$. *Then* $L^{\lambda}f(z)$ *belongs to the same class* $\mathcal{S}(1/2)$ *at least for* $\lambda\geqq\lambda_0$ *where* λ_0 *is a certain number less than* 3.

Proof. Putting $\alpha=1/2$ in (2.5) and using (1.5), with the aid of Lemma 1 and (2.7), we obtain

$$\begin{aligned}(2.8)\qquad &\sum_{n=2}^{\infty}\left(\frac{3n-1}{2}\right)\frac{|a_n|}{n^\lambda}\\ &=\frac{5|a_2|}{2^{\lambda+1}}+\frac{4|a_3|}{3^\lambda}+\frac{1}{2}\sum_{n=4}^{\infty}(3n-1)\frac{|a_n|}{n^\lambda}\\ &\leqq\frac{5}{2^{\lambda+1}}+\frac{1}{3^{\lambda-1}}+\frac{3}{4}\sum_{n=4}^{\infty}\frac{3n-1}{n^\lambda(n-1)}\\ &<\frac{5}{2^{\lambda+1}}+\frac{1}{3^{\lambda-1}}+\frac{3}{4}\sum_{n=4}^{\infty}\frac{1}{n^{\lambda-1}}\\ &<\frac{5}{2^{\lambda+1}}+\frac{1}{3^{\lambda-1}}+\frac{3}{4}\left\{\frac{2^{\lambda-2}}{2^{\lambda-2}-1}-\left(1+\frac{1}{2^{\lambda-1}}+\frac{1}{3^{\lambda-1}}\right)\right\}\\ &=\frac{1}{2^\lambda}+\frac{1}{4\cdot 3^{\lambda-1}}+\frac{3}{4(2^{\lambda-2}-1)}\\ &<1\end{aligned}$$

for any real $\lambda\geqq 3$. Thus we can show that $L^\lambda f(z)\in\mathcal{S}(1/2)$ by using Lemma 2.

Finally we can give the following problem.

PROBLEM. Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{S}(\alpha)$ with $0<\alpha\leqq 1$. Then does $L^\lambda f(z)$ belong to the same class $\mathcal{S}(\alpha)$ at least for $\lambda\geqq\lambda_0$ where λ_0 is a certain number less than 3?

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