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GENERIC SUBMANIFOLDS WITH NORMAL STRUCTURE OF AN ODD-DIMENSIONAL SPHERE (II)

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1. Introduction

Many authors studied the so-called generic submanifolds of a Riemannian manifolds and gave many valuable suggestions as the results ([2], [3], [5], [8]). And recently, the present authors studied generic submanifolds of an odd-dimensional sphere under the condition that the induced structure on the submanifold is normal and partially integrable. The purpose of the present paper is to characterize Einstein generic submanifolds of an odd-dimensional sphere tangent to the Sasakian structure vector field and compact generic submanifolds.

In characterizing the generic submanifolds, we will make use of the following theorems:

THEOREM A ([6]). *Let M be an n-dimensional complete generic submanifold with flat normal connection of an odd-dimensional unit sphere $S^{2m+1}(1)$ and let the Sasakian structure vector defined on $S^{2m+1}(1)$ be tangent to M. If the structure induced on M is normal and if the mean curvature vector of M is parallel in the normal bundle, then M is a pythagorean product of the form*

$$S^{p_1}(r_1)\times\cdots\times S^{p_N}(r_N)$$

where $p_1, \ldots, p_N$ are odd numbers $\geqq 1$, $r_1{}^2+\cdots+r_N{}^2=1$, $N=2m+2-n$.

THEOREM B ([6]). *Let M be an n-dimensional complete minimal generic submanifold with flat normal connection of an odd-dimensional unit sphere $S^{2m+1}(1)$ and let the Sasakian structure vector defined on $S^{2m+1}(1)$ be tangent to M. If the structure induced on M is normal, then M is a great sphere of $S^{2m+1}(1)$ or a pythagorean product of the form*

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$$S^{p_1}(r_1)\times\cdots\times S^{p_N}(r_N),$$

$p_1, \dots, p_N \geqq 1, p_1+\cdots+p_N=n, 1<N\leqq 2m-n+2$ *with essential codimension* $N-1$, *where* $r_t=\sqrt{p_t/n}$ $(t=1, \dots, N)$.

Manifolds, submanifolds, geometric objects and mappings discussed in this paper are assumed to be differentiable and of C^∞. We use throughout this paper the systems of indices as follows:

$$k, j, i, h=1, 2, \dots, 2m+1; \quad a, b, c, d, e=1, 2, \dots, n,$$
$$x, y, z, u, v, w=1^*, 2^*, \dots, p^*, \quad n+p=2m+1.$$

The summation convention will be used with respect to those systems of indices.

2. Generic submanifolds of a Sasakian manifold

Let $\tilde{M}$ be a $(2m+1)$-dimensional Sasakian manifold covered by a system of coordinate neighborhoods $\{U\,;\,x^h\}$ and $(F_j{}^h, g_{ji}, F_i)$ the set of structure tensors of M. Then we have

$$F_i{}^tF_t{}^h=-\delta_i^h+F_iF^h, \quad F_tF_i{}^t=0, \quad F_t{}^hF^t=0, F_tF^t=1 \tag{2.1}$$

and

$$F_j{}^tF_i{}^sg_{ts}=g_{ji}-F_jF_i, \tag{2.2}$$

F^h being the vector field associated with F_i, that is, $F^h=F_ig^{ih}$, g^{ih} being contravariant metric tensor of M. We also have

$$\nabla_jF^i=F_j{}^i \tag{2.3}$$

and

$$\nabla_jF_i{}^h=-g_{ji}F^h+\delta_j^hF_i, \tag{2.4}$$

where ∇_j denotes the operator of covariant differentiation with respect to the Christoffel symbols formed with g_{ji}.

Let M be an n-dimensional Riemannian manifold covered by a system of coordinate neighborhoods $\{V : y^a\}$ and isometrically immersed in M by the immersion $i : M\rightarrow\tilde{M}$. We identify $i(M)$ with M and represent the immersion by $x^h=x^h(y^a)$.

If we put $B_b{}^h=\partial_bx^h(\partial_b=\partial/\partial y^b)$, then $B_b{}^h$ are n linearly independent vectors of M tangent to M. Denoting by g_{cb} the Riemannian metric tensor of M, we have $g_{cb}=g_{ji}B_c{}^jB_b{}^i$ since the immersion is isometric.

If we denote by $C_x{}^h$ $2m+1-n$ mutually orthogonal unit normals to M, then we have $g_{ji}B_c{}^jC_x{}^i=0$ and the metric tensor of the normal bundle of M is given by $g_{zy}=g_{ji}C_z{}^jC_y{}^i=\delta_{zy}$, δ_{zy} being the Kronecker delta. Therefore, denoting by ∇_b the operator of van der Waerden-Bortolotti covariant differentiation with respect to the Christoffel symbols $\left\{{a \atop c\,b}\right\}$ formed with g_{cb}, we have equations of Causs and Weingarten for M

(2.5) $$\nabla_c B_b{}^h = h_{cb}{}^x C_x{}^h,$$

(2.6) $$\nabla_c C_x{}^h = -h_c{}^a{}_x B_a{}^h$$

respectively, where $h_{cb}{}^x$ are the second fundamental tensors with respect to the normals $C_x{}^h$ and $h_c{}^a{}_x = h_{cb}{}^y g^{ba} g_{yx}$, $(g^{ba})=(g_{ba})^{-1}$.

Denoting by $K_{kji}{}^h$, $K_{dcb}{}^a$ and $K_{dcy}{}^x$ the curvature tensors of Gauss, Codazzi and Ricci respectively

(2.7) $$K_{dcb}{}^a = K_{kji}{}^h B_d{}^k B_c{}^j B_b{}^i B^a{}_h + h_d{}^a{}_x h_{cb}{}^x - h_c{}^a{}_x h_{db}{}^x,$$

(2.8) $$0 = K_{kji}{}^h B_d{}^k B_c{}^j B_b{}^i C^x{}_h - (\nabla_d h_{cb}{}^x - \nabla_c h_{db}{}^x),$$

(2.9) $$K_{dcy}{}^x = K_{kji}{}^h B_d{}^k B_c{}^j C_y{}^i C^x{}_h + h_{de}{}^x h_c{}^e{}_y - h_{ce}{}^x h_d{}^e{}_y,$$

where we have put $B^a{}_h = B_b{}^j g^{ba} g_{jh}$, $C^x{}_h = C_y{}^j g^{yx} g_{jh}$ and $(g^{yx})=(g_{yx})^{-1}$.

An n-dimensional submanifold M is called a *generic* (an anti-holomorphic) submanifolds of the Sasakian manifold $\tilde{M}$ if M satisfies

$$N_P(M) \perp F(N_P(M))$$

at each point $P\in M$, where $N_P(M)$ denotes the normal space at P.

From now on, we consider generic submanifolds immersed in a $(2m+1)$-dimensional Sasakian manifold $\tilde{M}$. Then we can put in each coordinate neighborhood

(2.10) $$F_i{}^h B_b{}^i = f_b{}^a B_a{}^h - f_b{}^x C_x{}^h,$$

(2.11) $$F_i{}^h C_x{}^i = f_x{}^a B_a{}^h,$$

where $f_b{}^a$ is a tensor field of type (1, 1) defined on M, $f_c{}^x$ a local 1-form for each fixed index x and $f_x{}^a = f_c{}^y g^{ca} g_{yx}$. Also, we can put the Sasakian structure vector F^h of the form

(2.12) $$F^h = f^a B_a{}^h + f^x C_x{}^h,$$

f^a and f^x being vector fields defined on M and the normal bundle of M respectively.

Transvecting (2.10) and (2.11) with $F_h{}^k$ respectively and using (2.1), (2.10), (2.11) and (2.12), we find

$$f_b^e f_e^a = -\delta_b^a + f_b f^a + f_b^x f_x^a, \tag{2.13}$$

$$f_x^e f_e^y = \delta_x^y - f_x f^y, \tag{2.14}$$

$$f_b^e f_e^x = -f_b f^x, \quad f_x^e f_e^a = f_x f^a, \tag{2.15}$$

where f_c and f_x are 1-forms associated with f^a and f^x respectively, that is, $f_c = f^a g_{ac}$ and $f_x = f^y g_{yx}$.

Also, transvecting (2.12) with $F_h{}^k$ and using (2.1), (2.10) and (2.11), we get

$$f_e^a f^e + f_x^a f^x = 0, \tag{2.16}$$

$$f_e^x f^e = 0. \tag{2.17}$$

From (2.12), we have

$$f_a f^a + f_x f^x = 1 \tag{2.18}$$

with the help of $F_t F^t = 1$.

If we put $f_{cb} = f_c^a g_{ba}$, $f_{cx} = f_c^y g_{yx}$ and $f_{xc} = f_x^a g_{ca}$, then we can easily verify that

$$f_{cb} = -f_{bc}, \quad f_{xc} = f_{cx}.$$

If we apply the operator ∇_c of covariant differentiation to (2.10) and take account of (2.4), (2.5) and (2.6), then we have

$$(-g_{ji}F^h + \delta_j^h F_i) B_c^j B_b^i + h_{cb}{}^x f_x^a B_a^h$$
$$= (\nabla_c f_b^a) B_a^h + h_{ca}{}^x f_b^a C_x^h - (\nabla_c f_b^x) C_x^h + h_c{}^a{}_x f_b^x B_a^h,$$

which implies

$$\nabla_c f_b^a = -g_{cb} f^a + \delta_c^a f_b + h_{cb}{}^x f_x^a - h_c{}^a{}_x f_b^x, \tag{2.19}$$

$$\nabla_c f_b^x = g_{cb} f^x + h_{ce}{}^x f_b^e. \tag{2.20}$$

By the same way we can also obtain from (2.11),

$$\nabla_c f_x^a = \delta_c^a f_x - h_c{}^e{}_x f_e^a, \tag{2.21}$$

$$h_c{}^e{}_x f_e^y = h_{ce}{}^y f_x^e. \tag{2.22}$$

Differentiating (2.12) covariantly and using (1.3), we find

$$F_j{}^h B_c^j = (\nabla_c f^a) B_a^j + h_{ca}{}^x f^a C_x^h + (\nabla_c f^x) C_x^h - h_c{}^a{}_x B_a^h,$$

from which, taking account of (2.10),

$$\nabla_c f^a = f_c^a + h_c{}^a{}_x f^x \tag{2.23}$$

$$\nabla_c f^x = -f_c^x - h_{ce}{}^x f^e. \tag{2.24}$$

The mean curvature vector of M given by $H^h=1/nh^xC_x{}^h$ which is globally defined on M is said to be *parallel* in the normal bundle if $\nabla_c h^x=0$, where $h^x=g^{cb}h_{cb}{}^x$.

The induced structure $(f_c{}^a, g_{cb}, f_c{}^x, f_c, f^x)$ satisfying (2.13)～(2.18) is said to be normal if

$$h_c{}^e{}_x f_e{}^a - f_c{}^e h_e{}^a{}_x = 0$$

or, equivalently

$$h_{ce}{}^x f_a{}^e + h_{ae}{}^x f_c{}^e = 0. \tag{2.25}$$

Suppose that the ambient manifold is an odd-dimensional unit sphere $S^{2m+1}(1)$. Then, equations of Gauss, Codazzi and Ricci are respectively given by

$$K_{dcb}{}^a = \delta_d{}^a g_{cb} - \delta_c{}^a g_{db} + h_d{}^a{}_x h_{cb}{}^x - h_c{}^a{}_x h_{db}{}^x, \tag{2.26}$$

$$\nabla_d h_{cb}{}^x - \nabla_c h_{db}{}^x = 0, \tag{2.27}$$

$$K_{dcy}{}^x = h_{de}{}^x h_c{}^e{}_y - h_{ce}{}^x h_d{}^e{}_y, \tag{2.28}$$

since the ambient manifold $S^{2m+1}(1)$ is a space of constant curvature 1.

3. Generic submanifolds of an odd-dimensional sphere whose Sasakian structure vector is tangent to the submanifolds

In this section, we consider that M is an n-dimensional generic submanifold of S^{2m+1} (1) with the Sasakian structure vector F^h given by (2.12) tangent to M, that is, $f^x=0$. Then (2.13)～(2.24) reduce to

$$f_c{}^e f_e{}^a = -\delta_c{}^a + f_c f^a + f_c{}^y f_y{}^a, \tag{3.1}$$

$$f_c{}^e f_e{}^x = 0, \tag{3.2}$$

$$f_e{}^x f^e = 0, \tag{3.3}$$

$$f_e{}^a f^e = 0 \tag{3.4}$$

$$f_x{}^e f_e{}^y = \delta_x{}^y, \tag{3.5}$$

$$f_e f^e = 1, \tag{3.6}$$

$$\nabla_c f_b{}^a = -g_{cb} f^a + \delta_c{}^a f_b + h_{cb}{}^x f_x{}^a - h_c{}^a{}_x f_b{}^x, \tag{3.7}$$

$$\nabla_c f_b{}^x = h_{ce}{}^x f_b{}^e, \quad \nabla_c f_x{}^a = -h_c{}^e{}_x f_e{}^a, \tag{3.8}$$

$$h_c{}^e{}_x f_e{}^y = h_{ce}{}^y f_x{}^e, \tag{3.9}$$

$$\nabla_c f^a = f_c{}^a \tag{3.10}$$

$$f_c{}^x + h_{ce}{}^x f^e = 0. \tag{3.11}$$

From (3.1) and (3.2), we can easily find that M admits the so-called f-structure satisfying $f^3+f=0$.

The present authors proved

LEMMA 3.1 ([3]). *Let M be an n-dimensional generic submanifold of an odd-dimensional unit sphere $S^{2m+1}(1)$ with flat normal connection. If the structure induced on M such that the Sasakian structure vector of $S^{2m+1}(1)$ be tangent to M is normal, then we have*

$$h_{ce}{}^{x}h_{b}{}^{e}{}_{y}=P_{yz}{}^{x}h_{cb}{}^{z}+\delta_{y}{}^{x}g_{cb}, \tag{3.12}$$

$$\nabla_{c}h^{x}=\nabla_{c}P^{x}, \tag{3.13}$$

where we have put

$$P_{yz}{}^{x}=h_{cb}{}^{x}f_{y}{}^{c}f_{z}{}^{b}, \tag{3.14}$$

$$P^{x}=g^{yz}P_{yz}{}^{x}. \tag{3.15}$$

Proof. Transvecting (2.25) with $f_y{}^c$ and using (3.2), we find

$$h_{ce}{}^{x}f_{y}{}^{c}f_{b}{}^{e}=0,$$

from which, transvecting $f_a{}^b$ and making use of (3.1),

$$h_{ce}{}^{x}f_{y}{}^{e}=P_{yw}{}^{x}f_{c}{}^{w}-\delta_{y}{}^{x}f_{c} \tag{3.16}$$

with the help of (3.5), (3.11) and (3.14).

Putting $P_{zyx}=P_{zy}{}^{w}g_{wx}$, it is easily verified that P_{yzx} is symmetric for all indices because of (3.9) and (3.14).

If we transvect (3.16) with $h_b{}^c{}_z$ and make use of (3.11) and (3.16), then we obtain

$$h_{b}{}^{c}{}_{z}h_{ce}{}^{x}f_{y}{}^{e}=P_{yw}{}^{x}P_{zv}{}^{w}f_{b}{}^{v}-P_{yz}{}^{x}f_{b}+\delta_{y}{}^{x}f_{bz},$$

from which, taking account of the fact that the normal connection of M is flat and using (3.11) and (3.16),

$$P_{yz}{}^{w}P_{wv}{}^{x}f_{b}{}^{v}+g_{yz}f_{b}{}^{x}=P_{yw}{}^{x}P_{zv}{}^{w}f_{b}{}^{v}+\delta_{y}{}^{x}f_{bz},$$

or, transvecting $f_u{}^b$ and taking account of (3.5).

$$P_{yz}{}^{w}P_{wu}{}^{x}+g_{yzu}{}^{x}=P_{yw}{}^{x}P_{zu}{}^{w}+\delta_{y}{}^{x}g_{uz} \tag{3.17}$$

because of P_{yzx} is symmetric for all indices. Thus, it follows that

$$P_{zxw}P_{y}{}^{wx}=P_{x}P_{zy}{}^{x}+(p-1)g_{zy}. \tag{3.18}$$

Differentiating (3.16) covariantly along M and substituting (3.8) and (3.10), we gat

$$(\nabla_{d}h_{ce}{}^{x})f_{y}{}^{e}+h_{c}{}^{ex}h_{day}f_{e}{}^{a}=(\nabla_{d}P_{yz}{}^{x})f_{d}{}^{z}+P_{yz}{}^{x}h_{de}{}^{z}f_{c}{}^{e}-\delta_{y}{}^{x}f_{dc},$$

from which, taking the skew-symmetric part with respect to d and c and using (2.27),

$$(3.19)\quad -2h_c{}^{ex}h_{eay}f_d{}^a=(\nabla_d P_{yz}{}^x)f_c{}^z-(\nabla_c P_{yz}{}^x)f_d{}^z-2P_{yz}{}^x h_{ce}{}^z f_d{}^e-2\delta_y^x f_{dc}$$

with the help of (2.25) and (2.28) with $K_{dcy}{}^x=0$. Transvecting (3.19) with $f_w{}^d$ and using (3.2) and (3.5), we get

$$(3.20)\quad \nabla_c P_{yw}{}^x=(f_w{}^e\nabla_e P_{yz}{}^x)f_c{}^z,$$

which and $P_{yz}{}^x=P_{zy}{}^x$ imply

$$(\nabla_c P_{yz}{}^x)f_b{}^z=(f_y{}^e\nabla_e P_{wz}{}^x)f_c{}^z f_b{}^w.$$

Therefore (3.19) becomes

$$h_c{}^{ex}h_{eay}f_d{}^a=P_{yz}{}^x h_{ce}{}^z f_d{}^e+\delta_y^x f_{dc}.$$

If we apply $f_b{}^d$ to this and use (3.1), we obtain

$$\begin{aligned}&-h_c{}^{ex}h_{bey}+P_{yz}{}^w P_{vw}{}^x f_c{}^v f_b{}^z+f_c{}^x f_{by}\\&=-P_{yz}{}^x h_{cb}{}^z+P_{yw}{}^x P_{vz}{}^w f_b{}^z f_c{}^v-\delta_y^x g_{cb}+\delta_y^x f_c{}^z f_{bz}\end{aligned}$$

with the help of (3.11), or, take account of (3.17), we can find (3.12).

In the next place, differentiation (2.25) covariantly and substitution (3.7) yield

$$\begin{aligned}&(\nabla_d h_{ce}{}^x)f_b{}^e+h_{ce}{}^x(-g_{db}f^e+\delta_d^e f_b+h_{db}{}^y f_y{}^e-h_d{}^e{}_y f_b{}^y)+(\nabla_d h_{be}{}^x)f_c{}^e\\&\quad+h_{be}{}^x(-g_{dc}f^e+\delta_d^e f_c+h_{dc}{}^y f_y{}^e-h_d{}^e{}_y f_c{}^y)=0,\end{aligned}$$

from which, taking account of (3.11), (3.12) and (3.16),

$$(\nabla_d h_{ce}{}^x)f_b{}^e+(\nabla_d h_{be}{}^x)f_c{}^e=0,$$

or, taking the skew-symmetric part with respect to the indices d and c, and using (2.27),

$$(\nabla_d h_{be}{}^x)f_c{}^e-(\nabla_c h_{be}{}^x)f_d{}^e=0.$$

The last equations give $(\nabla_d h_{ce}{}^x)f_b{}^e=0$ by means of (2.27). Transvection $f_a{}^b$ yields

$$\nabla_d h_{ca}{}^x=(\nabla_d h_{ce}{}^x)f_y{}^e f_a{}^y+(\nabla_d h_{ce}{}^x)f^e f_a$$

with the help of (3.1), which implies

$$(3.21)\quad \nabla_d h^x=(\nabla_d h_{ce}{}^x)f_y{}^e f^{cy}+(\nabla_d h_{ce}{}^x)f^c f^e.$$

But, we see from (3.3) and (3.11) that $h_{ce}{}^x f^c f^e=0$. Differentiating this covariantly and making use of (3.4) and (3.10), we get $(\nabla_d h_{ce}{}^x) f^c f^e=0$. Consequently (3.21) becomes

(3.22) $$\nabla_d h^x=(\nabla_d h_{ce}{}^x) f_y{}^e f^{cy}.$$

On the other hand, we have from (3.14)

$$P^x=h_{ce}{}^e f_y{}^e f^{cy}.$$

If we differentiate this covariantly and take account of (3.8), we find

$$\nabla_d P^x=(\nabla_d h_{ce}{}^x) f_y{}^e f^{cy}+2h_c{}^{ex} h_{day} f_e{}^a f^{cy},$$

which means

$$\nabla_d P^x=(\nabla_d h_{ce}{}^x) f_y{}^e f^{cy}$$

with the help of (3.5) and (3.11). Thus, this together with (3.22) gives (3.13). Thus, Lemma 3.1 is completely proved.

LEMMA 3.2. *Under the same assumptions as those stated in* Lemma 3.1, *we have*

(3.23) $$(\nabla_c P_z)\{(h^w-P^w)P_{vw}{}^x+2(n-m-1)\delta_v{}^x\}=0.$$

Proof. Contraction (3.20) with respect to the indices y and x gives

(3.24) $$\nabla_c P_w=(f_w{}^e \nabla_e P_z) f_c{}^z$$

Transvecting this with $f_a{}^c$ and making use of (3.2), we find

(3.25) $$f_a{}^e \nabla_e P_w=0.$$

Also, transvection (3.24) with f^c yields

(3.26) $$f^e \nabla_e P_w=0$$

becase of (3.3),

Differentiating (3.24) covariantly and substituting (3.8), we get

$$\nabla_b \nabla_c P_w=(f_w{}^e \nabla_b \nabla_e P_z) f_c{}^z+(f_w{}^e \nabla_e P_z) h_{ba}{}^z f_c{}^a$$

because of (3.25), from which, taking the skew-symmetric part with respect to the induces b and c and making use of (2.25),

(3.27) $$(f_w{}^e \nabla_b \nabla_e P_z) f_c{}^z-(f_w{}^e \nabla_c \nabla_e P_z) f_b{}^z+2(f_w{}^e \nabla_e P_z) h_{ba}{}^z f_c{}^a=0$$

because the normal connection of M is flat.

Transvection $f_y{}^c$ implies

(3.28) $$f_w{}^e \nabla_b \nabla_e P_y=(f_w{}^e f_y{}^c \nabla_c \nabla_e P_z) f_b{}^z$$

with the help of (3.5). From which, taking the skew-symmetric part with respect to w and y and taking account of the fact that the normal connection is flat

(3.29) $$f_w{}^e\nabla_b\nabla_e P_y = f_y{}^e\nabla_b\nabla_e P_w.$$

If we substitute (3.29) into (3.27), then we obtain

$$(f_w{}^e\nabla_e P_z)h_{ba}{}^x f_c{}^a = 0,$$

from which, transvecting $f_d{}^w$ and using (3.1),

(3.30) $$(\nabla_d P_z)h_{be}{}^x f_c{}^e = 0$$

because of (3.25) and (3.26). If we transvect (3.30) with $f_a{}^c$ and use (3.1), then we have

(3.31) $$(\nabla_d P_z)(h_{ba}{}^x - P_{yw}{}^x f_b{}^w f_a{}^y + f_b f_a{}^x + f_a f_b{}^x) = 0$$

with the help of (3.11) and (3.16). Transvection $h^{ba}{}_v$ gives

$$(\nabla_d P_z)(P_{vw}{}^x h^w - P_{yw}{}^x P_v{}^{yw} + (n-2)\delta_v{}^x) = 0$$

because of (3.5), (3.11) and (3.12). The last relationship becomes

(3.32) $$(\nabla_d P_z)\{P_{vw}{}^x h^w - P_{vw}{}^x P^w + 2(n-m-1)\delta_v{}^x\} = 0$$

with the help of (3.18) and the fact that $p=2m+1-n$. This means the proof of our lemma.

4. Einstein generic submanifolds of an odd-dimensional sphere

In this section, we study the so-called Einstein generic submanifold M of an odd-dimensional sphere admitting the Ricci tensor K_{cb} of the form $K_{cb}=\frac{K}{n}g_{cb}$, where K is the scalar curvature of M. In particular, M is said to be proper Einstein if $K \neq 0$.

We have from (2.26)

(4.1) $$K_{cb} = (n-1)g_{cb} + h^x h_{cbx} - h_c{}^a{}_x h_{ba}{}^x,$$

which implies

(4.2) $$K = n(n-1) + h_x h^x - h_{cb}{}^x h^{cb}{}_x.$$

If the Sasakian structure vector F^h defined on $S^{2m+1}(1)$ is tangent to M, then we can write (4.1) and (4.2) respectively as follows:

(4.3) $$K_{cb} = 2(n-m-1)g_{cb} + (h_x - P_x)h_{cb}{}^x,$$

$$(4.4) \qquad K=2n(n-m-1)+(h^x-P^x)h_x$$

because of (3.12). Since M is of Einstein, we get from (4.3)

$$(4.5) \qquad 2(n-m-1)g_{cb}+(h^x-P^x)h_{cbx}=\frac{K}{n}g_{cb}.$$

Transvection $f_y{}^c f_z{}^b$ produces

$$(4.6) \qquad (h^x-P^x)P_{yzx}+2(n-m-1)g_{yz}=\frac{K}{n}g_{yz}$$

because of (3.5). Therefore, if M is a proper Einstein space, we have from (3.23) that $\nabla_c P_z=0$ and hence $\nabla_c h_z=0$ because of (3.13), that is, the mean curvature vector is parallel in the normal bundle. Thus, we have by means of Theorem A in the section 1.

THEOREM 4.1. *Let M be an n-dimensional complete proper Einstein generic submanifold with flat normal connection of an odd-dimensional unit sphere S^{2m+1} (1) and let the Sasakian structure vector defined on S^{2m+1} (1) be tangent to M. If the induced structure on M is normal, then M is a pythagorean product of the form*

$$S^{p_1}(r_1)\times\cdots\times S^{p_N}(r_N),$$

where $p_1,\ldots,p_N$ are odd numbers$\geqq 1$, $r_1{}^2+\cdots+r_N{}^2=1$, $N=2m+2-n$.

THEOREM 4.2. *Let M be an n-dimensional complete locally irreducible generic submanifold with flat normal connection of an odd-dimensional sphere $S^{2m+1}(1)$ and let the Sasakian structure vector defined on $S^{2m+1}(1)$ be tangent to M. If the induced structure on M is normal and the square of the length of the Ricci tensor is constant, then we have the same conclusion as that of* Theorem 4.1.

Proof. Transvecting (3.31) with g^{ba} and using (3.3), we get

$$(\nabla_d P_x)(h^x-P^x)=0,$$

or, using (3.13),

$$(\nabla_d h_x)(h^x-P^x)=0.$$

This fact together with (3.13) implies the scalar curvature K given by (4.4) is a constant.

From the Ricci identity and the fact that the normal connection is flat:

$$\nabla_d\nabla_c h_{ba}{}^x-\nabla_c\nabla_d h_{ba}{}^x=-K_{dcb}{}^e h_{ea}{}^x-K_{dca}{}^e h_{be}{}^x,$$

we have by contracting with respect to d and b

$$\nabla^d\nabla_d h_{ca}{}^x - \nabla_c\nabla_a h^x = K_{ce}h_a{}^{ex} - K_{dcae}h^{dex}$$

with the help of (2.27). Substituting (2.26) and (4.3) into the right hand side of this equation and taking account of (3.12), we obtain

$$\nabla^d\nabla_d h_{ca}{}^x - \nabla_c\nabla_a h^x = 0. \tag{4.7}$$

We now apply the operator $\nabla^d\nabla_d$ to (4.3) and make use of (3.13) and (4.7). Then we have

$$\nabla^d\nabla_d K_{cb} = (h^x - P^x)\nabla^d\nabla_d h_{xcb},$$

which becomes

$$\nabla^d\nabla_d K_{cb} = \nabla^d\nabla_d K$$

because of (4.4). On the other hand, K being constant, we find

$$\nabla^d\nabla_d K_{cb} = 0. \tag{4.8}$$

Therefore, the identity:

$$\frac{1}{2}\Delta(K_{cb}K^{cb}) = (\nabla^d\nabla_d K_{cb})K^{cb} + \|\nabla_d K_{cb}\|^2 \tag{4.9}$$

gives $\nabla_d K_{cb} = 0$ because of the fact that $K_{cb}K^{cb}$ is a constant, where Δ is the Laplacian given by $\Delta = g^{cb}\nabla_c\nabla_b$. Hence the Ricci tensor K_{cb} has the form

$$K_{cb} = \frac{K}{n}g_{cb} \quad (K \neq 0)$$

since M is locally irreducible, which indicates M is proper Einstein. Thus, Theorem 4.1 gives our assertion.

Replacing the condition the square of the length of the Ricci tensor being constant in Theorem 4.2 by the compactness of M, we see from (4.8) and (4.9) that the Ricci tensor is parallel. Thus we have

COROLLARY 4.3. *Let M be an n-dimensional compact orientable locally irreducible generic submanifold with flat normal connection of an odd-dimensional sphere $S^{2m+1}(1)$ tangent to the Sasakian structure vector. If the structure induced on M is normal, then M is the same type as that of* Theorem 4.1.

5. Compact generic submanifolds of S^{2m+1} (1)

First of all, we prove

LEMMA 5.1. *Let M be a compact orientable n-dimensional generic submanifold of* $S^{2m+1}(1)$. *Then we have*

$$(5.1)\quad \int_M \{\frac{1}{2}\|h_{ce}{}^x f_b{}^e + h_{be}{}^x f_c{}^e\|^2 + np - \|h_{cb}{}^x - \sqrt{n+2} f^x g_{cb}\|^2 + \|\nabla_c f^x\|^2 + h^z h_{cez} f^{ex} f_x{}^c + K_{dcy}{}^x f^{dy} f_x{}^c\} *1 = 0,$$

where $*1$ *denotes the volume element of M.*

Proof. From the Ricci identity for $f_b{}^x$:

$$\nabla_d \nabla_c f_b{}^x - \nabla_c \nabla_d f_b{}^x = -K_{dcb}{}^a f_a{}^x + K_{dcy}{}^x f_b{}^y,$$

we have

$$\nabla^b \nabla_c f_b{}^x = n \nabla_c f^c + K_{ce} f^{ex} + K_{dcy}{}^x f^{dy}$$

with the help of (2.20), or substitute (4.1),

$$(5.2)\quad \nabla^b \nabla_c f_b{}^x = n \nabla_c f^x + (n-1) f_c{}^x + h^z h_{cez} f^{ex} - h_{ca}{}^z h_e{}^a{}_z f^{ex} + K_{dcy}{}^x f^{dy}.$$

By means of (2.20) we have

$$(5.3)\quad \frac{1}{2}\|\nabla_c f_b{}^x + \nabla_b f_c{}^x - 2 g_{cb} f^x\|^2 = (\nabla_c f_{bx})(\nabla^b f^{cx}) + \|\nabla_c f_b{}^x\|^2 - 2n f_x f^x.$$

Since we see from (2.13) and (2.20) that

$$\|\nabla_c f_b{}^x\|^2 = n f_x f^x + h_{cb}{}^x h^{cb}{}_x - (h_{ce}{}^x f^{ze})(h_d{}^c{}_x f_z{}^d) - (h_{ce}{}^x f^e)(h_d{}^c{}_x f^d),$$

(5.3) reduces to

$$(5.4)\quad (\nabla_c f_b{}^x)(\nabla^b f_x{}^c) = \frac{1}{2}\|h_{ce}{}^x f_b{}^e + h_{be}{}^x f_c{}^e\|^2 + n f_x f^x - h_{cb}{}^x h^{cb}{}_x + (h_{ce}{}^x f^{ez})(h_d{}^c{}_x f_z{}^d) + (h_{ce}{}^x f^e)(h_d{}^c{}_x f^d).$$

with the help of (2.19).

Substituting (5.2) and (5.4) into the identity:

$$\nabla^b\{f_x{}^c(\nabla_c f_b{}^x)\} = (\nabla_c f_{bx})(\nabla^b f^{cx}) + f^{cx} \nabla^b \nabla_c f_{bx},$$

we find

$$(5.5)\quad \nabla^b\{f_x{}^c(\nabla_c f_b{}^x)\} = \frac{1}{2}\|h_{ce}{}^x f_b{}^e + h_{ce}{}^x f_b{}^e\|^2 + n f_x f^x - h_{cb}{}^x h^{cb}{}_x + (h_{ce}{}^x f^e)(h_d{}^c{}_x f^d) + n f^{cx} \nabla_c f_x + (n-1)(p - f_x f^x) + h^z h_{cez} f^{ex} f_x{}^c + K_{dcy}{}^x f^{dy} f_x{}^c$$

because of (2.14).

On the other hand, we have from (2.24) that

$$(h_{ce}{}^x f^e)(h_d{}^c{}_x f^d) = p - f_x f^x + \|\nabla_c f^x\|^2 + 2 f^{cx} \nabla_c f^x$$

with the help of (2.14). Substituting this into the identity:

$$\nabla^c (f_x f_c{}^x) = f^{cx} (\nabla_c f_x) + n f_x f^x \tag{5.6}$$

which with (2.20) implies

$$\begin{aligned}&(h_{ce}{}^x f^e)(h_d{}^c{}_x f^d) + n f^{cx} \nabla_c f_x \\ &= p - f_x f^x - n(n+2) f_x f^x + \|\nabla_c f^x\|^2 + (n+2) \nabla^c (f_x f_c{}^x).\end{aligned}$$

Using this, (5.5) reduces to

$$\begin{aligned}&\nabla^b (f_x{}^c \nabla_c f_b{}^x) - (n+2) \nabla^c (f_x f_c{}^x) \\ &= \frac{1}{2} \|h_{ce}{}^x f_b{}^e + h_{be}{}^x f_c{}^e\|^2 - h_{cb}{}^x h^{cb}{}_x - n(n+2) f_x f^x + \|\nabla_c f^x\|^2 \\ &\qquad + h^z h_{cez} f^{ex} f_x{}^c + K_{dcy}{}^x f^{dy} f_x{}^c.\end{aligned} \tag{5.7}$$

Since we have

$$\|h_{cb}{}^x - \sqrt{n+2} f^x g_{cb}\|^2 = h_{cb}{}^x h^{cb}{}_x + n(n+2) f_x f^x - 2\sqrt{n+2}\, \nabla^e f_e$$

because of (2.23), (5.7) becomes

$$\begin{aligned}&\nabla^c \{ f_x{}^b \nabla_b f_c{}^x - (n+2) f_x f_c{}^x + 2\sqrt{n+2} f_c \} \\ &= \frac{1}{2} \|h_{ce}{}^x f_b{}^e + h_{be}{}^x f_c{}^e\|^2 + np - \|h_{cb}{}^x - \sqrt{n+2} f^x g_{cb}\|^2 \\ &\qquad + \|\nabla_c f^x\|^2 + h^z h_{cez} f^{ex} f_x{}^c + K_{dcy}{}^x f^{dy} f_x{}^c.\end{aligned}$$

Since M is compact orientable, we have (5.1).
From (2.25), (5.1) and (5.6) we have

LEMMA 5.2. *Let M be a compact orientable n-dimensional generic submanifold of $S^{2m+1}(1)$ with flat normal connection. Suppose that M is minimal or the second fundamental form $h_{cb}{}^x$ of M is positive semi-definite. If*

$$\|h_{cb}{}^x - \sqrt{n+2} f^x g_{cb}\|^2 \leqq np$$

at every point of M, where $p = 2m - n + 1$, then the Sasakian structure vector defined on $S^{2m+1}(1)$ is tangent to M, that is, $f^x = 0$, and the structure induced on M is normal.

According to Theorem A and B in the section 1 and Lemma 5.2, we conclude the followings;

THEOREM 5.3. *Let M be a compact orientable n-dimensional generic* *submanifold of an odd-dimensional sphere* S^{2m+1} (1) *with flat normal connection such that the second fundamental form is positive semi-definite. If the mean curvature vector is parallel in the normal bundle and satisfies one of the followings:*

(1) $\|h_{cb}{}^x - \sqrt{n+2}\, f^x g_{cb}\|^2 \leqq n(2m-n+1)$ *at every point of* M

(2) $\|h_{cb}{}^x\|^2 \leqq n(2m-n+1)$ *at every point of* M *and the Sasakian structure defined on* S^{2m+1} (1) *is tangent to* M. *Then* M *is a pythagorean product of the form*

$$S^{p_1}(r_1) \times \cdots \times S^{p_N}(r_N),$$

where $p_1, \ldots, p_N$ *are odd numbers* $\geqq 1$, $r_1{}^2 + \cdots + r^2{}_N = 1$, $N = 2m-n+2$.

THEOREM 5.4. *Let* M *be a compact orientable* n*-dimensional generic submanifold of an odd-dimensional sphere* S^{2m+1} (1) *with flat normal connection. If* (1) *or* (2) *in* Theorem 5.3 *holds, then* M *is a great sphere* S^n (1) *or a pythagorean product of the form*

$$S^{p_1}(r_1) \times \cdots \times S^{p_N}(r_N),$$

where $p_1, \ldots, p_N \geqq 1$, $p_1 + \cdots + p_N = n$, $1 < N \leqq 2m-n+2$, *in this case* M *is of essential codimension* N-1 *and* $r_t = \sqrt{p_t/n}$ $(t = 1, 2, \ldots, N)$.

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