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PRIMARY IDEALS

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In the theory of commutative rings with identity R one usually defines a primary ideal to be an ideal I of R such that if $xy\in I$ and $x\notin I$, then $y^n\in I$ for some integer $n\geq 1$. The following types of ideals in arbitrary rings with identity are closely related to primary ideals in the commutative case and seem to have many of the properties one would want primary ideals in general rings to have. In this note we work exclusively with left unitary R-modules. It is clear that one has similar results for right R-modules. We give the definitions and prove several theorems and propositions to illustrate the point made above.

DEFINITION. Suppose R is a ring. An ideal I of R is *(left) primary* if there is a faithful indecomposable R/I-module M. If the module M can in addition be chosen such that M is both Artinian and Noetherian, then I is a *strongly (left) primary* ideal.

If R is a commutative ring (with identity), then we have families F_1. F_2 and F_3, where: F_1 is the collection of all strongly primary ideals of R, F_2 is the collection of all primary ideals of R and F_3 is the collection of all (left) primary ideals of R.

THEOREM 1. *If R is a commutative ring and if the families F_1, F_2 and F_3 are as defined, then we have inclusions: $F_1\subseteq F_2\subseteq F_3$.*

Proof. Suppose I is a strongly primary ideal of R, and suppose M is a faithful indecomposable Artinian and Noetherian R/I-module.

Suppose that $xy\in I$, with $x\notin I$ and $y^n\notin I$ for all integers $n\geq 1$. Given $r\in R, r$ induces an R/I-endomorphism of M by left multiplication of elements of M by $\bar{r}=r+I$ when R is commutative. Now, by Fitting's Lemma $M=\bar{y}^nM\oplus P$ for some integer $n\geq 1$ and thus since $y^n\notin I$ and since M is indecomposable $M=\bar{y}^nM=\bar{y}M$. But then it follows that

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$\bar{x}M=\bar{x}\bar{y}M=0$, contradicting the fact that M is a faithful R/I-module. Hence I is a primary ideal of R.

Now suppose that I is a primary ideal of R and suppose that we consider $R/I=M$ as a module over itself. M is a faithful R/I-module. If M is not indecomposable, $M=L_1\oplus L_2$, where the L_i are ideals of R/I. Hence $\bar{1}=e_1+e_2$, with $L_i=Me_i$, while e_1 and e_2 are a pair of orthogonal idempotents. If $\bar{x}=e_1$ and $\bar{y}=e_2$, then $xy\in I$, $x\notin I$, $y\notin I$, contradicting the fact that I is a primary ideal. Thus M is indecomposable and I is a left primary ideal.

Thus $F_1\subseteq F_2\subseteq F_3$ as claimed.

Using the Krull-Schmidt Theorem we have some immediate results on intersections. Suppose that I is an ideal of R such that there is a faithful R/I-module M which is both Artinian and Noetherian. We shall refer to such an ideal as *a Krull-Schmidt ideal* of R. The module M may not be unique, but it does yield an intersection theorem for Krull-Schmidt ideals in terms of strongly primary ideals.

THEOREM 2. *Let I be a Krull-Schmidt ideal of the ring R with associated faithful Artinian and Noetherian R/I-module M. Then M determines I as the unique intersection of a finite number of strongly primary ideals of R.*

Proof. If $M=M_1\oplus\cdots\oplus M_r=N_1\oplus\cdots\oplus N_s$, where the modules M_i and N_j are indecomposable for all i and j, then by the Krull-Schmidt theorem $r=s$, and $M_i\cong N_i$ for all i, without loss of generality.

Now, since the isomorphism is also R-isomorphism, it follows that if I_i is the annihilator of M_i in R and if J_i is the annihilator of N_i in R, then $I_i=J_i$.

Since M is a faithful R/I-module, it follows that $I=I_1\cap I_2\cap\cdots\cap I_r$. Since each M_i is a faithful indecomposable R/I_i-module, which is Artinian and Noetherian, it follows that I_i is a strongly primary ideal of R.

If $I_i\subseteq I_j$ for some i and j, then we may adjust M by removing the module M_j altogether. Having done so, we have removed the 'embedded component' I_j in the intersection. Continuing this we may always take M such that $I=I_1\cap\cdots\cap I_r$ is an irredundant intersection.

The following propositions are offered in the nature of observations and are proven easily.

PROPOSITION 1. *If R is a ring and if I is a primary ideal such that R/I has finite characteristic m, then $m=p^n$ for some prime p and some integer $n\geq 1$.*

PROPOSITION 2. *If P is a left primitive ideal of R, then P is a strongly primary ideal of R.*

PROPOSITION 3. *If I is an ideal such that R/I has no divisors of zero, then I is a primary ideal of R.*

A Steinitz ring is a local ring with a radical which is either left or right vanishing, i. e., given a sequence $\{x_i\}_{i=1}^{\infty}$ of elements in the radical, there is an index n such that $x_n \cdot x_{n-1} \cdot \ldots \cdot x_1 = 0$ (left vanishing) or $x_1 \cdot x_2 \cdot \ldots \cdot x_n = 0$ (right vanishing). The quotient rings R/I, where I is a strongly primary ideal have centers which are subrings of Steinitz rings.

THEOREM 3. *Let I be a strongly primary ideal of the ring R and let M be an associated faithful indecomposable Artinian and Noetherian R/I -module M. Then $E=\mathrm{Hom}_{R/I}(M, M)$ is a right and left Steinitz ring and if C is the center of R/I, there is a natural inclusion $C\subseteq E$.*

Proof. By Fitting's Lemma it follows that E is a local ring whose radical is a nil ideal. We map the elements of C to the corresponding left multiplications on M. Since M is a faithful module, this mapping is an injection. Since left multiplication by elements in C yields a collection of elements in E, the conclusion follows provided we demonstrate that E is a Steinitz ring.

Suppose $N\neq 0$, and N is a submodule of M. If $\phi(N)=N$ for $\phi\in E$, then $\phi^k(N)=N$ and ϕ is not nilpotent. Hence ϕ is a unit of E. Suppose that $\{\phi_i\}_{i=1}^{\infty}$ is a sequence of elements in the radical of E. Then $M\supset \phi_1(M)\supset\phi_2\phi_1(M)\supset\cdots$ is a properly descending chain of submodules, and thus since M is Artinian, there is an integer n such that $\phi_n\phi_{n-1}\cdots \phi_1(M)=0$. Hence, $\phi_n\cdots\phi_1=0$ in E, and E is a left Steinitz ring. Also $M\supseteq\phi_1(M)\supseteq\phi_1\phi_2(M)\supseteq\phi_1\phi_2\phi_3(M)\supseteq\cdots$ implies that there is an integer n such that $\phi_1\phi_2\phi_3\cdots\phi_n(M)=\phi_1\phi_2\phi_3\cdots\phi_n\phi_{n+1}(M)$. Then $\phi_1\phi_2\cdots\phi_n(1-\phi_{n+1})(M)=0$. Since $1-\phi_{n+1}$ is a unit of E, $(1-\phi_{n+1})(M)=M$, and $\phi_1\phi_2\phi_n M=0$. Hence $\phi_1\phi_2\ldots\phi_n=0$ and E is a right Steinitz ring also.

COROLLARY 1. *The radical J of the ring E in* Theorem 3 *is nilpotent.*

Proof. Consider the following sequence of sub-modules of M,

$$JM \geq J^2M \geq J^3M \geq J^4M \geq \cdots$$

since all these are R/I modules and since M is Artinian, there is an integer n such that $J^nM = J^{n+1}M$. If $J^nM \neq 0$, we have a map f from J^nM to J^nM and a map g of J^mM to J such that

$$m = g(m) \cdot f(m).$$

If we define $m_1 = m$ for some nonzero element m of J^nM and $m_{i+1} = f(m_i)$ then $g(m_1)g(m_2)g(m_3)g(m_i)m_i = m_1$ for each i. This is a contradiction because J is a Steinitz ring.

COROLLARY 2. *Let I be a Krull-Schmidt ideal of the ring R. Let C be the center of R/I. Then C is a subring of a finite direct sum of Steinitz rings.*

Proof. Let $M = M_1 \oplus \ldots \oplus M_r$ and let $I = I_1 \cap \ldots \cap I_r$ be as in the proof of Theorem 2. Inject C into $E_1 \oplus \ldots \oplus E_r$, where $E_i = \mathrm{Hom}_{R/I_i}(M_i, M_i)$ is a Steinitz ring by Theorem 3. The corollary follows.

Since the center of a Steinitz ring is a Steinitz ring, and since the ring C is in the center of the ring E via the embedding given in Theorem 3, we may require in addition that C can be included in acommutative Steinitz ring (resp. a finite direct sum of commutative Steinitz rings).

Proofs of proposition 4 and theorem 4 have been provided for completeness, although they follow rather directly.

PROPOSITION 4. *Suppose M is a faithful indecomposable R-module. Suppose that R_n is the complete ring of $n \times n$ matrices with coefficients in R, and suppose that N is the R_n-module with elements $(m_1, \ldots, m_n)$, $m_i \in M$, addition componentwise and R_n-action given by*

$$(r_{ij})(m_1, \ldots, m_n) = (w_1, \ldots, w_n)$$

were

$$w_i = \sum_{j=1}^{n} r_{ij} m_j.$$

Then N is a faithful indecomposable R_n-module.

Proof. If $(r_{ij})N = 0$, then

$$(r_{ij})(m_1, 0, \ldots, 0) = (r_{11}m_1, \ldots, r_{n1}m_1) = 0,$$

implies $r_{11}=r_{21}=\cdots=r_{n1}=0$.

Similarly, $r_{ij}=0$ for all i and j, and thus N is a faithful R_n-module.

If A is an R_n-submodule of N and if $(m_1, \ldots, m_n)\in A$, then for any permutation π of $(1, \ldots, n)$ we also have $(m_{\pi(1)}, \ldots, m_{\pi(n)})\in A$. Thus if B is the collection of first components of elements of A, then B is an R-submodule of M. Furthermore, if $N=A_1\oplus A_2$, and if B_i is associated with A_i, then $M=B_1\oplus B_2$. It follows that if M is indecomposable then N is indecomposable as an R_n-module and conversely.

If N is an R_n-module, let subgroups N_i be defined by $N_i=E_{ii}N$, where E_{ii} is the matrix with 1 in position (i, i) and 0's elsewhere. Then $I=E_{11}+\cdots+E_{nn}$, yields a decomposition $N=N_1\oplus\cdots\oplus N_n$ of N as a direct sum of abelian groups. The groups N_i are all isomorphic as abelian groups, since we may map N_i to N_j via permutation matrices. Define an R-action on N_i via $r\cdot n=rE_{ii}n$ for $n\in N_i$. Now, if P is a permutation matrix, then $(rI)Pn=P\cdot rIn=Pr\cdot n$, and so the R-modules N_i are also R-isomorphic. Let $M=N_1=\cdots=N_n$, and construct the R_n-module associated with M as in proposition 4. It follows that M and N are isomorphic R_n-modules.

THEOREM 4. *Suppose that R is a ring and I is an ideal of R. Let R_n be the complete ring of $n\times n$ matrices with coefficients in R. Then I_n is an ideal of R. Conversely, if J is any ideal of R_n, then $J=I_n$ for some ideal I of R. If I is a primary ideal then I_n is a primary ideal. If I is a strongly primary ideal then I_n is a strongly primary ideal. The converses of the last two statements are also true.*

Proof. If M is a faithful indecomposable R/I module, then there exists a faithful indecomposable $(R/I)_n$-module N as constructed in the proof of proposition 4. Conversely, if N is a faithful indecomposable $(R/I)_n$-module, then there exists a faithful indecomposable R/I-module M as constructed in the discussion following proposition 4. Hence I is a primary ideal if and only if I_n is a primary ideal. Since the R-module M is both Artinian and Noetherian if and only if the associated R_n-module N is both Artinian and Noetherian, if follows that I is strongly primary if and only if I_n is strongly primary.

We close with a last result analogous to the usual results in the case of commutative rings.

THEOREM 5. *Let R be a ring. Suppose that* $x^n \neq 0$. *Then there is a primary ideal I such that* $x \notin I$.

Proof. Let P be a left ideal of R which is maximal with respect to being disjoint from the set $\{x^n\}^{\omega}_{n=1}$. Consider the left R-module $M=(P+Rx)/P$, and let I be the annihilator of the R-module M. Then M is a faithful R/I-module.

Suppose that $M=A\oplus B$, and let L_1 and L_2 be the complete inverse images in R of A and B respectively. Then $L_1+L_2=P+Rx$, where both L_1 and L_2 are left ideals containing P, while $L_1\cap L_2=P$. If $L_1\neq P$ and $L_2\neq P$, then for some integer n, $x^n\in L_1\cap L_2=P$, a contradiction. Thus M is an indecomposable R/I-module, and I is a primary ideal.

Now if $x\in I$, then $x(P+Rx)\subseteq P$, and thus $x^3\in P$, a contradiction. Hence $x\notin I$, and the theorem follows.

COROLLARY 1. *If I is the intersection of all primary ideals of the ring R, then I is a nil ideal.*

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