J. Korean Math. Soc. **21** (1984), No. 2. p. 121-140.

ON CHARACTERS OF η-RELATED TENSORS IN COSYMPLECTIC AND SASAKIAN MANIFOLDS (2)*

SANG-SEUP EUM

0. Introduction

A $(2n+1)$-dimensional differentiable manifold M is called to have a *cosymplectic structure* if there is given a positive definite Riemannian metric g_{ji} and a triplet $(\varphi_k{}^j, \xi^j, \eta_k)$ of $(1,1)$ type tensor field $\varphi_k{}^j$, vector field ξ^j and 1-form η_k in M which satisfy the following equations

$$(0.1)\qquad \varphi_j{}^i\varphi_i{}^h=-\gamma_j{}^h,\quad \varphi_j{}^i\xi^j=0,\quad \eta_i\varphi_j{}^i=0,\quad \eta_t\xi^t=1,$$
$$g_{st}\varphi_j{}^s\varphi_i{}^t=\gamma_{ji},\quad \eta_i=g_{ih}\xi^h,$$

where

$$(0.2)\qquad \gamma_{ji}=g_{ji}-\eta_j\eta_i,\quad \gamma_j{}^h=g^{ht}\gamma_{jt}$$

and

$$(0.3)\qquad \nabla_k\varphi_j{}^i=0,\quad \nabla_k\eta_j=0,$$

where ∇_k indicates the covariant differentiation with respect to g_{ji}. By virtue of the last equation of (0.1), we shall write η^h instead of ξ^h in the sequel. The indices $h, i, j, k, \ldots$ run over the range $\{1, 2, \ldots, 2n+1\}$.

In the present paper, we define an *η-projective vector field* v^h in a cosymplectic manifold M by the condition

$$(0.4)\qquad \mathcal{L}_v\left\{{h \atop ji}\right\}=\nabla_j\nabla_i v^h+v^t K_{tji}{}^h=p_j\gamma_i{}^h+p_i\gamma_j{}^h$$

for a certain covector field p_i, where $\left\{{h \atop ji}\right\}$, $K_{tji}{}^h$ and $\mathcal{L}_v$ are the Christoffel symbols formed with g_{ji}, the curvature tensor and the Lie derivation with respect to v^i on M respectively.

The purpose of the present paper is to investigate the properties of η-projective vector fields in a compact cosymplectic manifold.

---

*) This reseach is supported by Korea Science and Engineering Foundation Research Grant 1983.

Received December 1, 1983.

1. η-projective vector fields in a cosymplectic manifold

In a $(2n+1)$-dimensional cosymplectic manifold M with the cosymplectic structure (φ, η, g), we easily obtain the following relations.

$$K_{kji}{}^{t}\eta_t=0, \quad K_{ji}\eta^i=0 \tag{1.1}$$

by virtue of the Ricci identity with respect to η^i, where K_{ji} is the Ricci tensor of M. Moreover using the Ricci identity with respect to $\varphi_i{}^h$, we easily see that $K_{jt}\varphi_i{}^t+K_{tjis}\varphi^{ts}=0$ and from which,

$$K_{jt}\varphi_i{}^t+K_{it}\varphi_j{}^t=0. \tag{1.2}$$

Since

$$K_{tjis}\varphi^{ts}=\frac{1}{2}(K_{tjis}-K_{sjit})\varphi^{ts}=-\frac{1}{2}K_{tsji}\varphi^{ts}, \tag{1.3}$$

we obtain

$$\varphi^{ts}K_{tsj}{}^{h}=2K_{jt}\varphi^{ht}. \tag{1.4}$$

In a previous paper (Eum, [1]), we proved that if M is acosymplectic manifold of constant curvature with respect to γ_{ji}, then the curvature tensor of M is of the form:

$$K_{kji}{}^{h}=\frac{K}{2n(2n-1)}(\gamma_k{}^h\gamma_{ji}-\gamma_j{}^h\gamma_{ki}), \tag{1.5}$$

K being the constant scalar curvature.

If we substitute (1.5) into (0.4), we obtain

$$\nabla_k\nabla_j v^h+\frac{K}{2n(2n-1)}v^t(\gamma_t{}^h\gamma_{kj}-\gamma_k{}^h\gamma_{tj})=\gamma_k{}^h p_j+\gamma_j{}^h p_k. \tag{1.6}$$

In this place, we consider a system of partial differential equation

$$\nabla_k\nabla_j p_h+\frac{K}{2n(2n-1)}(2\gamma_{jh}p_k+\gamma_{kh}p_j+\gamma_{kj}p_h)=0 \tag{1.7}$$

which is obtained by the substitution into (1.6) of

$$v^h=-\frac{n(2n-1)}{K}p^h. \tag{1.8}$$

The integrability condition of (1.7) is given by

$$\begin{aligned}&\nabla_s\nabla_l(\nabla_k\nabla_j p_h)-\nabla_l\nabla_s(\nabla_k\nabla_j p_h)\\&=-K_{slk}{}^{t}\nabla_t\nabla_j p_h-K_{slj}{}^{t}\nabla_k\nabla_t p_h-K_{slh}{}^{t}\nabla_k\nabla_j p_t.\end{aligned} \tag{1.9}$$

If we assume that $p_t\eta^t=0$, then the condition (1.9) is satisfied by

(1.5) and (1.7). In this case we obtain

$$(1.10)\qquad \mathcal{L}_p \begin{Bmatrix} h \\ kj \end{Bmatrix} = \nabla_k \nabla_j p^h + p^t K_{tkj}{}^h = -\frac{K}{n(2n-1)}(\gamma_k{}^h p_j + \gamma_j{}^h p_k)$$

by virtue of (1.5) and (1.7), where $\mathcal{L}_p$ indicates the Lie derivation with respect to p^h.

Thus we have the following

THEOREM 1.1. *Let M be a cosymplectic manifold of constant curvature with respect to γ_{ji}. In this case, if p^i in M belongs to the distribution orthogonal to η^i, that is, $p_t\eta^t=0$, then p^i is an η-projective vector field locally and the associated vector of p^i is given by $-\frac{K}{n(2n-1)}p^i$, K being the constant scalar curvature.*

By contractions in (0.4), find

$$(1.11)\qquad \nabla_j \nabla_t v^t = (2n+1)p_j - (p_t\eta^t)\eta_j$$

and

$$(1.12)\qquad \nabla_t \nabla_i v^t = v^t K_{ti} + (2n+1)p_i - (p_t\eta^t)\eta_i.$$

Transvecting (0.4) with η_h and taking account of (0.2), (0.3) and (1.1), we easily see that

$$(1.13)\qquad \nabla_j \nabla_i (v^t\eta_t) = 0.$$

Therefore in a compact orientable cosymplectic manifold M, we obtain

$$(1.14)\qquad v^t\eta_t = c$$

c being a constant (Yano, [5]), and from which

$$(1.15)\qquad \mathcal{L}_v \eta_i = 0.$$

Substituting (0.4) into the formula (Yano [5])

$$(\mathrm{I})\qquad \mathcal{L}_v K_{kji}{}^h = \nabla_k \mathcal{L}_v \begin{Bmatrix} h \\ ji \end{Bmatrix} - \nabla_j \mathcal{L}_v \begin{Bmatrix} h \\ ki \end{Bmatrix}$$

we obtain

$$(1.16)\qquad \mathcal{L}_v K_{kji}{}^h = (\nabla_k p_j - \nabla_j p_k)\gamma_i{}^h + (\nabla_k p_i)\gamma_j{}^h - (\nabla_j p_i)\gamma_k{}^h,$$

and from which

$$(1.17)\qquad \mathcal{L}_v K_{ji} = \nabla_i p_j - 2n\nabla_j p_i - \eta^t\{(\nabla_t p_j)\eta_i + (\nabla_t p_i)\eta_j\} + \eta^t(\nabla_j p_t)\eta_i.$$

Using the relation $\mathcal{L}_v K_{ji} = \mathcal{L}_v K_{ij}$, we obtain

(1.18) $$(2n+1)(\nabla_i p_j - \nabla_j p_i) = \eta^t\{(\nabla_i p_t)\eta_j - (\nabla_j p_t)\eta_i\}.$$

Transvecting (1.18) with η^i, we obtain

(1.19) $$(2n+1)\eta^t\nabla_t p_j - 2n\nabla_j(p_t\eta^t) = \mu\eta_j,$$

where we have put

(1.20) $$\mu = \eta^i(\nabla_i p_t)\eta^t.$$

We consider on the case that M is a compact cosymplectic manifold. Taking account of the second equation of (1.1) and (1.15), we obtain

(1.21) $$\eta_h \pounds_v K_k{}^h = \eta_h \pounds_v (K_{kt}g^{th}) = 0.$$

Substituting (1.17) into (1.21), we obtain

(1.22) $$(2n-1)\nabla_k(p_t\eta^t) = \eta_h K_{kt}\pounds_v g^{th}$$

by virtue of (1.15).

Transvecting (1.22) with η^k and taking account of (1.1) and (1.20), we see that

(1.23) $$\mu = 0.$$

Substituting (1.23) into (1.19), we obtain

(1.24) $$(2n+1)\eta^t\nabla_t p_j = 2n\nabla_j\rho,$$

where we have put

(1.25) $$\rho = p_t\eta^t.$$

Substituting (1.16) into the equation

$$\varphi^{ts}\{(\pounds_v K_{tsj}{}^h)\eta^j + K_{tsj}{}^h\pounds_v\eta^j\} = 0,$$

which is obtained from the first equation of (1.1), and taking account of (0.1), (1.25) and the fact that $\varphi^{ts} = -\varphi^{st}$, we obtain

(1.26) $$\varphi^{ts}K_{tsj}{}^h\pounds_v\eta^j = 2\varphi^{ht}\nabla_t\rho.$$

Substituting (1.17) into the equation which is obtained from the second equation of (1.1)

(1.27) $$2\varphi^{ht}\{(\pounds_v K_{jt})\eta^j + K_{jt}\pounds_v\eta^j\} = 0$$

and taking account of (1.23) and (1.25), we obtain

(1.28) $$2\varphi^{ht}K_{jt}\pounds_v\eta^j = 2(2n-1)\varphi^{ht}\nabla_t\rho.$$

Taking account of (1.3), (1.26) and (1.28), we obtain

$$(n-1)\varphi^{ht}\nabla_t\rho=0. \tag{1.29}$$

Transvecting (1.29) with φ_{hk} and taking account of the fact that $\eta^t\nabla_t\rho=\mu=0$, we obtain

$$(n-1)\nabla_t\rho=0. \tag{1.30}$$

Thus we see that if $n>1$, then ρ=constant.

On the other hand, if we transvect (1.11) with η^j, then we have $\nabla_j(v\eta^j)=2n\rho$, where we have put $v=\nabla_t v^t$. Then by the Green's theorem, we see that if $n>1$, then

$$\rho=0. \tag{1.31}$$

Next, we investigate on the case of $n=1$, that is, $2n+1=3$. It is well known that the conformal curvature tensor of Weyl vanishes identically in a 3-dimensional Riemannian manifold. Therefore we have the following formula in the case of $n=1$:

$$K_{kji}{}^t+K_{ki}\delta_j^t-K_{ji}\delta_k^t+g_{ki}K_j^t-g_{ji}K_k^t-\frac{K}{2}(g_{ki}\delta_j^t-g_{ji}\delta_k^t)=0. \tag{1.32}$$

Transvecting (1.32) with $\eta_t\eta^j$, we obtain

$$K_{ki}=\frac{K}{2}\gamma_{ki} \tag{1.33}$$

by virtue of (1.1).

Substituting (1.33) into (1.32), we obtain

$$K_{kji}{}^h=\frac{K}{2}(\gamma_k^h\gamma_{ji}-\gamma_j^h\gamma_{ki}). \tag{1.34}$$

Thus we have the following (Eum, [2])

THEOREM 1.2. *A 3-dimensional cosymplectic manifold with constant scalar curvature K is a manifold of constant curvature with respect to* γ_{ji}.

In the case of $n=1$, we obtain

$$\nabla_k\rho=K_{kt}\pounds_v\eta^t \tag{1.35}$$

by virtue of (1.15) and (1.22).

Substituting (1.33) into (1.35) and taking account of the fact that $\eta_t\pounds_v\eta^t=0$, we see that

$$\nabla_k\rho=\frac{K}{2}g_{kt}\pounds_v\eta^t. \tag{1.36}$$

Substituting (0. 4) into the well known formula:

$$(\text{II}) \qquad \mathcal{L}_v(\nabla_j\eta^h)-\nabla_j(\mathcal{L}_v\eta^h)=(\mathcal{L}_v\left\{{h \atop ji}\right\})\eta^i,$$

we easily sea that

$$(1.37) \qquad -\nabla_j(\mathcal{L}_v\eta^h)=\rho\gamma_j{}^h.$$

and from which

$$(1.38) \qquad \nabla_t(\mathcal{L}_v\eta^t)=-2\rho$$

by virtue of the fact that $n=1$.

If the scalar curvature K is non-zero constant, then operating ∇_j to (1. 36) and taking account of (1. 37), we obtain

$$(1.39) \qquad \nabla_j\nabla_k\rho+\frac{K}{2}\rho\gamma_{jk}=0,$$

and from which

$$(1.40) \qquad \nabla^t\nabla_t\rho+K\rho=0.$$

Under the assumption that K is non-zero constant, if we take account of (1. 15), (1. 17) and (1. 33), then we obtain

$$(1.41) \quad \frac{K}{2}\mathcal{L}_v g_{ji}=\nabla_i p_j-2\nabla_j p_i-\eta^t\{(\nabla_t p_j)\eta_i+(\nabla_t p_i)\eta_j\}+(\nabla_j\rho)\eta_i,$$

where ρ is defined by (1. 25).

Substituting (1. 41) into the identity:

$$(\text{III}) \qquad \mathcal{L}_v\left\{{h \atop ji}\right\}=\frac{1}{2}g^{hk}(\nabla_j\mathcal{L}_v g_{ki}+\nabla_i\mathcal{L}_v g_{kj}-\nabla_k\mathcal{L}_v g_{ji}),$$

and transvect the result with η^i, we obtain

$$(1.42) \quad K\rho\gamma_{jh}=\eta^i\nabla_i\nabla_j p_h-2\eta^i\nabla_i\nabla_h p_j-\eta^t\{\eta^i(\nabla_i\nabla_t p_h)\eta_j+\eta^i(\nabla_i\nabla_t p_j)\eta_h\},$$

where we have used the relation

$$(1.43) \qquad \eta^j\nabla_j\nabla_k\rho=0$$

which is obtained from (1. 39).

Transvecting (1. 42) with g^{jh} and taking account of (1. 25) and (1. 43), we obtain

$$(1.44) \qquad 2K\rho=-\eta^i\nabla_i\nabla_t p^t.$$

Substituting $\nabla_i\nabla_t p^t=\nabla_t\nabla_i p^t-K_{it}p^t$ into (1. 44), we obtain

$$(1.45) \qquad 2K\rho=-\nabla_t(\eta^i\nabla_i p^t).$$

Substituting (1.24) into (1.45), we obtain

$$3K\rho=-\nabla_t\nabla^t\rho. \tag{1.46}$$

Comparing (1.40) with (1.46), we obtain in the case of $n=1$ also

$$\rho=0 \tag{1.47}$$

by virtue of the assumption $K\neq 0$.

Taking account of (1.31) and (1.47), we have the following

THEOREM 1.3. *If a compact cosympletic manifold M of dimension $2n+1$ $(n\geqq 1)$ admits an η-projective vector field v^i and the scalar curvature K of M is non-zero constant, then the associated vector p^i of v^i belongs to the distribution orthogonal to η^i, that is, $p_t\eta^t=0$.*

2. Lie derivations with respect to an η-projective vector in a cosymplectic manifold

In the present section, we calcuate the Lie derivations of some geometrical objects in the cosymplectic manifold M admitting an η-projective vector field v^i.

Substituting (1.31) into (1.11), we obtain

$$\nabla_j v=(2n+1)p_j, \tag{2.1}$$

where $v=\nabla_t v^t$. Thus p_i is a gradient vector.

Substituting the fact that $\nabla_j p_i=\nabla_i p_j$ into (1.16), we obtain

$$\pounds_v K_{kji}{}^h=(\nabla_k p_i)\gamma_j{}^h-(\nabla_j p_i)\gamma_k{}^h. \tag{2.2}$$

Substituting (1.23) and (1.31) into (1.19), we obtain

$$\eta^t\nabla_t p_j=0. \tag{2.3}$$

Thus we see from (1.17) that

$$\pounds_v K_{ji}=-(2n-1)\nabla_j p_i \tag{2.4}$$

by virtue of (1.31), (2.1) and (2.3).

Substituting (0.4) into the formula

$$\nabla_k\pounds_v g_{ji}=\nabla_k(\nabla_j v_i+\nabla_i v_j),$$

we obtain

$$\nabla_k\pounds_v g_{ji}=2p_k\gamma_{ji}+p_j\gamma_{ki}+p_i\gamma_{jk} \tag{2.5}$$

and from which

$$\nabla^k \mathfrak{L}_v g^{ji} = -2p^k\gamma^{ji} - p^j\gamma^{ki} - p^i\gamma^{jk}. \tag{2.6}$$

We defone a tensor field G_{ji} on M by

$$G_{ji} = K_{ji} - \frac{K}{2n}\gamma_{ji}, \tag{2.7}$$

where K is the scalar curvature of M, then we see easily that

$$G_{ji} = G_{ij}, \quad G_{ji}g^{ji} = G_i{}^i = 0, \quad \eta^t G_{jt} = 0.$$

Denoting the Lie derivation with respect to η^i by $\mathfrak{L}_\eta$ in M, we obtain

$$\mathfrak{L}_\eta \left\{ {h \atop ji} \right\} = \nabla_j \nabla_i \eta^h + \eta^t K_{tji}{}^h = 0$$

and from which

$$\mathfrak{L}_\eta K_{kji}{}^h = \eta^t \nabla_t K_{kji}{}^h = 0. \tag{2.8}$$

Contracting with respect to h and k, we obtain

$$\eta^t \nabla_t K_{ji} = 0. \tag{2.9}$$

We define an *η-projective curvature tensor* by

$$P_{kji}{}^h = K_{kji}{}^h - \frac{1}{2n-1}(\gamma_k{}^h K_{ji} - \gamma_j{}^h K_{ki}), \tag{2.10}$$

then this tensor field satisfies

$$P_{kji}{}^h = -P_{jki}{}^h, \quad P_{kjt}{}^t = 0, \quad P_{tji}{}^t = 0, \tag{2.11}$$
$$P_{kji}{}^h + P_{ikj}{}^h + P_{jik}{}^h = 0$$

and

$$P_{kji}{}^h g^{ji} = \frac{2n}{2n-1} G_k{}^h, \quad P_{kji}{}^h \eta_h = 0. \tag{2.12}$$

If the scalar curvature K is non-zero constant, then using

$$\nabla^j K_{ji} = \frac{1}{2}\nabla_i K = 0, \quad \nabla_t K_{kji}{}^t = \nabla_k K_{ji} - \nabla_j K_{ki}$$

and

$$\eta^k \nabla_k G_{ji} = 0 \tag{2.13}$$

which is obtained from (2.9), we see that

$$\nabla^k P_{kji}{}^h = \frac{2(n-1)}{2n-1}\nabla^h G_{ji} - \nabla_i G_j{}^h. \tag{2.14}$$

Calculating $\pounds_v G_{ji}$ and taking account of (2.4) and (2.7), we can see that

$$(2.15)\qquad \pounds_v G_{ji} = -(\nabla_j w_i + \nabla_i w_j)$$

if the scalar curvature is non-zero constant, where we have put

$$(2.16)\qquad w^h = \frac{2n-1}{2} p^h + \frac{K}{2n} v^h.$$

Substituting (2.2) and (2.4) into the Lie derivation of (2.10) and taking account of (1.15), we obtain

$$(2.17)\qquad \pounds_v P_{kji}{}^h = \frac{1}{2n-1}\left\{(\pounds_v \gamma_j{}^h) K_{ki} - (\pounds_v \gamma_k{}^h) K_{ji}\right\}$$
$$= \frac{1}{2n-1}\{\eta_j \eta^t (\nabla_t v^h) K_{ki} - \eta_k \eta^t (\nabla_t v^h) K_{ji}\}.$$

We assume that the scalar curvature K is non-zero constant. In this case, we obtain

$$(2.18)\qquad (\nabla^k \pounds_v P_{kji}{}^h) g^{ji} = 0$$

by virtue of (0.4) and (1.1).

From the first equation of (2.12), we obtain

$$(\pounds_v P_{kji}{}^h) g^{ji} + P_{kji}{}^h \pounds_v g^{ji} = \frac{2n}{2n-1} \pounds_v G_k{}^h$$

and from which

$$(2.19)\qquad (\nabla^k \pounds_v P_{kji}{}^h) g^{ji} + (\nabla^k P_{kji}{}^h) \pounds_v g^{ji} + P_{kji}{}^h (\nabla^k \pounds_v g^{ji})$$
$$= \frac{2n}{2n-1} \nabla^k \pounds_v G_k{}^h.$$

Substituting (2.6), (2.14) and (2.18) into (2.19), we obtain

$$(2.20)\qquad \left[\frac{2n}{2n-1} \nabla^k \pounds_v G_k{}^h - \left\{\frac{2(n-1)}{2n-1} \nabla^h G_{ji} - \nabla_i G_j{}^h\right\} \pounds_v g^{ji} + \frac{2n}{2n-1} G_j{}^h p^j\right] w_h = 0.$$

3. A decomposition of an η-projective vector in a compact cosymplectic manifold with non-zero constant scalar curvature

In the present section, we use for briefness the following notations:

$$(3.1)\qquad I_1 = \int_M G_{jh} p^j w^h dV, \qquad I_2 = \int_M (\nabla_h G_{ji})(\pounds_v g^{ji}) w^h dV,$$

$$I_3=\int_M (\nabla_i G_{jh})(\pounds_v g^{ji}) w^h dV,\quad I_4=\int_M (\nabla^k \pounds_v G_k{}^h) w_h dV,$$

where dV denotes the volume element of M, and

$$\alpha=(\nabla_i w^i)^2,\quad \beta=(\nabla_j w_i+\nabla_i w_j)(\nabla^j w^i+\nabla^i w^j). \tag{3.2}$$

Let M be a compact cosymplectic manifold with non-zero constant scalar curvature and let M admits an η-projective vector field v^i defined by (0.4). In this case, we shall calculate the values of the integrals (3.1) following same ways as the processes of [3].

By using the identity

$$\nabla_i \Delta p=\nabla^j\nabla_j p_i-K_{ji}p^j,$$

where $p_i=\nabla_i p$ and $\Delta=g^{ji}\nabla_j\nabla_i$, and taking account of (1.31) and (2.7), we obtain

$$I_1=-\int_M (\nabla_i\Delta p) w^i dV+\int_M (\nabla^j\nabla_j p_i) w^i dV-\frac{K}{2n}\int_M p_i w^i dV.$$

Taking account of (2.16) and the Green's theorem, we obtain

$$\begin{aligned}
-\int_M (\nabla_i\Delta p) w^i dV&=\int_M (\Delta p)(\nabla_i w^i) dV\\
&=\frac{2}{2n-1}\int_M \alpha dV-\frac{K}{n(2n-1)}\int_M (\nabla_t v^t)(\nabla_i w^i) dV\\
&=\frac{2}{2n-1}\int_M \alpha dV+\frac{K}{n(2n-1)}\int_M (\nabla_i\nabla_t v^t) w^i dV\\
&=\frac{2}{2n-1}\int_M \alpha dV+\frac{(2n+1)K}{n(2n-1)}\int_M p_i w^i dV.
\end{aligned}$$

Consequently, we have

$$I_1=\frac{2}{2n-1}\int_M \alpha dV+\int_M (\nabla^t\nabla_t p_i) w^i dV+\frac{(2n+3)K}{n(2n-1)}\int_M p_i w^i dV.$$

By using

$$\nabla^j(\nabla_j v_i+\nabla_i v_j)=\nabla^j \pounds_v g_{ji}=(2n+3)p_i$$

we see that

$$\begin{aligned}
&\int_M (\nabla^t\nabla_t p_i) w^i dV+\frac{(2n+3)K}{2n(2n-1)}\int_M p_i w^i dV\\
&=\frac{1}{2n-1}\left[\int_M \left\{\nabla^j\left(\frac{2n-1}{2}\nabla_j p_i+\frac{K}{2n}\nabla_j v_i\right)\right\} w^i dV\right.\\
&\left.+\int_M \left\{\nabla^j\left(\frac{2n-1}{2}\nabla_i p_j+\frac{K}{2n}\nabla_i v_j\right)\right\} w^i dV\right]
\end{aligned}$$

$$=\frac{1}{2n-1}\int_M \nabla^j(\nabla_j w_i+\nabla_i w_j)w^i dV$$

$$=\frac{1}{2n-1}\left[\int_M \nabla^j\{(\nabla_j w_i+\nabla_i w_j)w^i\}dV\right.$$

$$\left.-\int_M(\nabla_j w_i+\nabla_i w_j)\nabla^j w^i dV\right]$$

$$=-\frac{1}{2(2n-1)}\int_M \beta dV.$$

Substituting this equation into above equation, we obtain

$$I_1=\frac{2}{2n-1}\int_M \alpha dV-\frac{1}{2(2n-1)}\int_M \beta dV. \tag{3.3}$$

The integral I_2 is expressed by

$$I_2=\int_M \nabla_h\{G_{ji}(\pounds_v g^{ji})w^h\}dV-\int_M G_{ji}(\nabla_h \pounds_v g^{ji})w^h dV$$

$$-\int_M (G_{ji}\pounds_v g^{ji})\nabla_h w^h dV.$$

Substituting (2.6) into this equation, we obtain

$$I_2=2\int_M G_{ji}p^j w^i dV-\int_M\{\pounds_v(G_{ji}g^{ji})-g^{ji}\pounds_v G_{ji}\}\nabla_t w^t dV$$

$$=2I_1-\int_M g^{ji}(\nabla_j w_i+\nabla_i w_j)\nabla_t w^t dV.$$

Hence we get

$$I_2=2I_1-2\int_M \alpha dV. \tag{3.4}$$

Since

$$g^{kj}\pounds_v(\nabla_k G_{ji})=g^{kj}\left[\nabla_k\pounds_v G_{ji}-\left(\pounds_v\begin{Bmatrix}t\\kj\end{Bmatrix}\right)G_{ti}-\left(\pounds_v\begin{Bmatrix}t\\ki\end{Bmatrix}\right)G_{jt}\right] \tag{3.5}$$

$$=\nabla^j\pounds_v G_{ji}-3G_{ji}p^j,$$

and

$$\int_M(\nabla^j\pounds_v G_{ji})w^i dV=-\int_M\{\nabla^j(\nabla_j w_i+\nabla_i w_j)\}w^i dV=\frac{1}{2}\int_M \beta dV, \tag{3.6}$$

which is obtained from (2.15), the integral I_3 is expressed by

$$I_3=\int_M[\pounds_v\{\nabla_j G_{ih})g^{ji}\}]w^h dV-\int_M g^{ji}(\pounds_v\nabla_j G_{ih})w^h dV$$

$$=-\int_M(\nabla^j\pounds_v G_{ji})w^i dV+3\int_M G_{ji}p^j w^i dV.$$

Hence substituting (3.6) into this equation, we obtain

$$(3.7) \qquad I_3=-\frac{1}{2}\int_M \beta dV+3I_1.$$

Lastly, we calculate the integral I_4.

$$I_4=\int_M [\nabla^k \{\pounds_v(G_{kj}g^{jh})\}]w_h dV$$
$$=\int_M (\nabla^k \pounds_v G_{kj})w^j dV+\int_M \{\nabla^k(G_{kj}\pounds_v g^{jh})\}w_h dV.$$

Substituting (3.6) into this equation and taking account of (2.6), we obtain

$$I_4=\frac{1}{2}\int_M \beta dV-3\int_M G_{ji}p^j w^i dV=\frac{1}{2}\int_M \beta dV-3I_1.$$

Thus we have

$$(3.8) \qquad I_4=-I_3.$$

Integrating (2.20) over M, we obtain

$$(3.9) \qquad 2n(I_1+I_4)-2(n-1)I_2+(2n-1)I_3=0.$$

Substituting (3.3), (3.4), (3.7) and (3.8) into (3.9), we obtain

$$(3.10) \qquad 2(2n-3)\int_M \alpha dV+\int_M \beta dV=0.$$

In the case of $n>1$, since $\alpha>0$ and $\beta>0$ over M we find that

$$(3.11) \qquad \alpha=0, \qquad \beta=0.$$

That is, we obtain

$$(3.12) \qquad \nabla_t w^t=0, \qquad \nabla_j w_i+\nabla_i w_j=0.$$

Therefore w^h is a Killing vector.

We consider in the case of $n=1$. Since p_i is a gradient vector and K is non-zero constant, we obtain

$$(3.13) \qquad \eta^t \nabla_t p_i=0$$

by virtue of (1.19), (1.20) and (1.47)

Substituting (1.47) and (3.13) into (1.41), we obtain

$$(3.14) \qquad \frac{K}{2}\pounds_v g_{ji}=-\nabla_j p_i$$

and from which

(3.15) $$\frac{K}{2}(\nabla_j v_i+\nabla_i v_j)=-\nabla_j p_i.$$

(3.15) and the fact that $\nabla_j p_i=\nabla_i p_j$ shows that if we put

$$w^h=\frac{1}{2}p^h+\frac{K}{2}v^h,$$

then w^h is a Killing vector, that is,

(3.16) $$\nabla_j w_i+\nabla_i w_j=0.$$

Thus an η-projective vector v^h is decomposed in the form

(3.17) $$v^h=\frac{2n}{K}\left(w^h-\frac{2n-1}{2}p^h\right),$$

where w^h is a Killing vector and p_i is a gradient vector.

The uniqueness of this decomposition is proved by the following way. In fact if

$$w^h-\frac{2n-1}{2}p^h='w^h-\frac{2n-1}{2}'p^h,$$

then $p^h-'p^h$ also a Killing vector. On the other hand, since $p_i-'p_i$ is a gradient vector, we see that

$$\nabla_j(p_i-'p_i)=0,$$

and from which

$$\nabla_j\nabla_i(p-'p)=0$$

where we have put $\nabla_i p=p_i$, $\nabla_i 'p='p_i$.

Since M is compact and orientable, we see that $p-'p$ is a constant. (Yano, [5]) Thus we obtain $p_i='p_i$. Therefore the uniqueness of the decomposition is proved.

Substituting (2.16) into the second equation of (3.12), we obtain

$$(2n-1)\nabla_j p_i+\frac{K}{2n}(\nabla_j v_i+\nabla_i v_j)=0.$$

Operating ∇_k to this equation and taking account of (0.4), we obtain

(3.18) $$\nabla_k\nabla_j p_i+\frac{K}{2n(2n-1)}(2p_k\gamma_{ji}+p_j\gamma_{ki}+p_i\gamma_{kj})=0.$$

Transvecting (3.18) with g^{kj}, we see that

(3.19) $$\nabla^t\nabla_t p^h=-\frac{(2n+3)K}{2n(2n-1)}p^h.$$

Therefore $K>0$ since K is non-zero constant. (Yano, [5])

Taking account of (0.4) and (3.16), we easily obtain

$$\pounds_w\left\{{h \atop ji}\right\}=\frac{2n-1}{2}\pounds_p\left\{{h \atop ji}\right\}+\frac{K}{2n}\pounds_v\left\{{h \atop ji}\right\}, \tag{3.20}$$

where $\pounds_w$ and $\pounds_p$ indicate the Lie derivations with respect to w^h and p^h respectively.

Since w^h is a Killing vector, we see that

$$\pounds_p\left\{{h \atop ji}\right\}=-\frac{K}{n(2n-1)}\pounds_v\left\{{h \atop ji}\right\}$$

by virtue of the identity (III) of section 1 and (3.20). Thus we obtain

$$\pounds_p\left\{{h \atop ji}\right\}=-\frac{K}{n(2n-1)}(p_j\gamma_i^{\ h}+p_i\gamma_j^{\ h}).$$

Therefore, taking account of above results, we have the following

THEOREM 3.1. *Let M be a compact $(2n+1)$ $(n\geqq 1)$ dimensional cosymplectic manifold with non-zero constant scalar curvature K. If M admits an η-projective vector field v^h defined by* (0.4), *then $K>0$ and v^h is decomposed uniquely in the form:*

$$v^h=\frac{2n}{K}\left(w^h-\frac{2n-1}{2}p^h\right),$$

where w^h is a Killing vector field and p^h is the associated vector field of v^h. Moreover p^h is also an η-projective vector field and the associated vector field of p^h is proportional (with constant coefficient $-\frac{K}{n(2n-1)}$) to p^h itself.

Substituting (3.18) into the identity

$$\nabla_k\nabla_j p^h-\nabla_j\nabla_k p^h=K_{kjt}{}^h p^t,$$

we obtain

$$U_{kji}{}^h p_h=0, \tag{3.21}$$

whrre we have put

$$U_{kji}{}^h=K_{kji}{}^h-\frac{K}{2n(2n-1)}(\gamma_k^{\ h}\gamma_{ji}-\gamma_j^{\ h}\gamma_{ki}). \tag{3.22}$$

Since w^h is a Killing vector field, we obtain

$$\pounds_v g_{ji}=(\nabla_j v_i+\nabla_i v_j)=-\frac{2n(2n-1)}{K}\nabla_j p_i \tag{3.23}$$

by virtue of (2.16), and from which

$$(3.24) \qquad \pounds_v g^{ji} = \frac{2n(2n-1)}{K} \nabla^j p^i.$$

Sibstituting (3.24) into $\pounds_v \eta^h = \pounds_v (g^{hi}\eta_i)$, we obtain

$$(3.25) \qquad \pounds_v \eta^h = 0$$

by virtue of (1.15), (1.31) and (3.24).

From (3.25), we obtain

$$(3.26) \qquad \pounds_v \gamma_j{}^i = 0.$$

Taking account of (1.15), (2.2), (3.22), (3,23) and (3.26), we see that

$$(3.27) \qquad \pounds_v U_{kji}{}^h = 0.$$

If we substitute (3.27) into the identity:

$$\pounds_v \nabla_l U_{kji}{}^h - \nabla_l \pounds_v U_{kji}{}^h$$
$$= U_{kji}{}^t \pounds_v \begin{Bmatrix} h \\ lt \end{Bmatrix} - U_{tji}{}^h \pounds_v \begin{Bmatrix} t \\ lk \end{Bmatrix} - U_{kti}{}^h \pounds_v \begin{Bmatrix} t \\ lj \end{Bmatrix} - U_{kjt}{}^h \pounds_v \begin{Bmatrix} t \\ li \end{Bmatrix},$$

then, we obtain

$$(3.28) \qquad \pounds_v \nabla_l K_{kji}{}^h = -(2U_{kji}{}^h p_l + U_{lji}{}^h p_k + U_{kli}{}^h p_j + U_{kjl}{}^h p_i)$$

by virtue of (3.21).

Transvecting this equation with p^k and taking account of (3.21), we obtain

$$(3.29) \qquad (\pounds_v \nabla_l K_{kji}{}^h) p^k = -U_{lji}{}^h p_k p^k.$$

Contracting with respect to h and k in (3.28), we obtain

$$(3.30) \qquad \pounds_v \nabla_l K_{ji} = -G_{lj} p_i,$$

and from which

$$(3.31) \qquad (\pounds_v \nabla_l K_{ji}) p^i = -G_{lj} p_i p^i.$$

Thus by (3.21), (3.22), (3.29) and (3.31), we have the following

THEOREM 3.2. *Under the same assumption for M as the theorem* 3.1, *we have the following propositions.*

If one of the following two conditions is satisficd, then M is a cosymplectic manifold of constant curvature with respect to γ_{ji}.

(1) *The Lie algebra of all η-projective vectors is transitive in M.*

(2) *M is a symmetric manifold.*

Moreover if M is a manifold of Ricci parallel, then M is an η-*Einstein manifold.*

4. Hypersurfaces of a cosymplectic manifold admitting an η-projective vector field

We consider the distribution orthogonal to η^h in a $(2n+1)$-dimensional cosymplectic manifold M.

If X and Y are vectors contained in such a distribution, then $[X, Y]=\nabla_X Y-\nabla_Y X$ is also contained in such a distribution. Therefore by a theorem of Frobenius, such a $2n$-dimensional distributions is integrable. Moreover, such a distribution is evidently parallel. Therefore M is locally a product manifold of a $2n$-dimensional Riemannian manifold and a 1-dimensional Riemannian manifold. If M is complete and simply connected, then there exists a hypersurface M^{2n} of M such that

$$M=M^{2n}\times R^1, \tag{4.1}$$

η^h is normal to M^{2n} and M^{2n} is complete and simply connected.

Let the hypersurface M^{2n} is covered by a system of coordinate neighborhoods $\{V\,;\,y^a\}$, then M^{2n} is expresses by $x^h=x^h(y^a)$. Denoting $B_a{}^h=\partial_a x^h$, $(\partial_a=\partial/\partial y^a)$, the induced metric tensor g_{ba} on M^{2n} from that of M is given by $g_{ba}=B_b{}^j B_a{}^i g_{ji}$.

Taking account of the fact that $\nabla_b=B_b{}^j\nabla_j$, ∇_b being the operator of covariant differentiation with respect to g_{ba}, and the Weingarten's formula: $\nabla_b\eta^h=-h_b{}^a B_a{}^h$, h_{ba} being the second fundamental tensor of M^{2n}, we easily see that $h_{ba}=0$, that is, M^{2n} is a totally geodesic hypersurface of M. Therefore, the Gaussian equation for M^{2n} is given by

$$K_{dcba}=K_{hkji}B_d{}^h B_c{}^k B_b{}^j B_a{}^i, \tag{4.2}$$

where $K_{dcb}{}^a$ is the curvature tensor of M^{2n}.

We denote by $(B^a{}_h, \eta_h)$ the inverse matrix of the matrix $\begin{pmatrix} B_b{}^k \\ \eta^k \end{pmatrix}$.

In this case, an η-projective vector field v^h of M is decomposed in the form

$$v^h=B_a{}^h u^a+\alpha\cdot\eta^h, \tag{4.3}$$

where $u_b(=B_b{}^k v_k)$ is a covector field of M^{2n}.

Taking account of the fact that $h_{ba}=0$ and (4.2), we easily verify the following equation

$$(4.4)\qquad B_c{}^k B_b{}^j B^a{}_h(\nabla_k\nabla_j v^h + v^t K_{tkj}{}^h) = \nabla_c\nabla_b u^a + u^e K_{ecb}{}^a.$$

Substituting (0.4) into (4.4), we obtain

$$(4.5)\qquad \mathcal{L}_u\begin{Bmatrix} a \\ cb \end{Bmatrix} = \nabla_c\nabla_b u^a + u^e K_{ecb}{}^a = \delta_c{}^a t_b + \delta_b{}^a t_c,$$

where $t_c = B_c{}^k p_k$ and $\mathcal{L}_u$ denotes the Lie derivation with respect to u^a in M^{2n}. Thus we have the following

THEOREM 4.1. *Let M be a $(2n+1)$-dimensional complete and simply connected cosymplectic manifold. Then M is a product manifold of a totally geodesic hypersurface M^{2n} and a 1-dimensional Riemannian manifold R^1. If M admits an η-projective vector field v^h, then M^{2n} admits a projective vector field u^a.*

On the other hand, transvecting (3.18) with $B_c{}^k B_b{}^j B_a{}^i$, we obtain

$$(4.6)\qquad \nabla_c\nabla_b t_a + \frac{K}{2n(2n-1)}(2g_{ba}t_c + g_{ca}t_b + g_{cb}t_a) = 0,$$

where K is the constant scalar curvature of M and $t_a = \nabla_a p$.

Transvecting (4.2) with $g^{da}g^{cb}$, we easily verify the fact that the scalar curvature of M^{2n} is equal to the scalar curvature of M. Taking account of a theorem of Obata ([4]) and (4.6), we obtain the following (cf. Theorem A of [3])

THEOREM 4.2. *Let M be a $(2n+1)$-dimensional compact, connected and simply connected cosymplectic manifold with non-zero constant scalar curvature K. If M admits an η-projective vector field v^h then the hypersurface M^{2n} orthogonal to η^h is globally isometric to a sphere of radius $\sqrt{2n(2n-1)/K}$ in the Euclidean $(2n+1)$-space.*

5. An η-projective vector field in a Sasakian manifold

If a set (φ, η, g) of a tensor field φ of type (1, 1), a vector field η and a Riemannian metric tensor g satisfies (0.1), (0.2) and additionally

$$\varphi_{ji} = \frac{1}{2}(\partial_j\eta_i - \partial_i\eta_j)$$

then, such a set is called a *contact structure*. A manifold with a normal contact structure is called a *Sasakian manifold*.

It is well known that in a Sasakian manifold, the following equations are satisfied:

(5.1) $$\nabla_i\eta^h=\varphi_i^h,\quad \nabla_j\varphi_i^h=-g_{ji}\eta^h+\delta_j^h\eta_i,$$

(5.2) $$\eta_t K_{kji}{}^t=\eta_k g_{ji}-\eta_j g_{ki},$$

(5.3) $$K_{jt}\eta^t=2n\eta_j.$$

In the present section, we investigate an η-projective vector field v^h defined by

(5.4) $$\mathcal{L}_v\left\{{h \atop ji}\right\}=\nabla_j\nabla_i v^h+v^t K_{tji}{}^h=p_j\gamma_i{}^h+p_i\gamma_j{}^h$$

in a Sasakian manifold.

Differentiating (5.4) covariantly, we obtain

(5.5) $$\nabla_k\mathcal{L}_v\left\{{h \atop ji}\right\}=(\nabla_k p_j)\gamma_i{}^h+(\nabla_k p_i)\gamma_j{}^h \\ -p_j(\varphi_{ki}\eta^h+\varphi_k{}^h\eta_i)-p_i(\varphi_{kj}\eta^h+\varphi_k{}^h\eta_j).$$

Substituting (5.5) into the identity:

$$\mathcal{L}_v K_{kji}{}^h=\nabla_k\mathcal{L}_v\left\{{h \atop ji}\right\}-\nabla_j\mathcal{L}_v\left\{{h \atop ki}\right\},$$

we obtain

(5.6) $$\mathcal{L}_v K_{kji}{}^h=(\nabla_k p_j-\nabla_j p_k)\gamma_i{}^h+(\nabla_k p_i)\gamma_j{}^h-(\nabla_j p_i)\gamma_k{}^h \\ -p_j(\varphi_{ki}\eta^h+\varphi_k{}^h\eta_i)+p_k(\varphi_{ji}\eta^h+\varphi_j{}^h\eta_i) \\ -p_i(2\varphi_{kj}\eta^h+\varphi_k{}^h\eta_j-\varphi_j{}^h\eta_k).$$

Transvecting (5.6) with η_h, we find

(5.7) $$\eta_h\mathcal{L}_v K_{kji}{}^h=p_k\varphi_{ji}-p_j\varphi_{ki}-2p_i\varphi_{kj}.$$

Taking the Lie derivation of the both sides of (5.2), we obtain

$$\eta_t\mathcal{L}_v K_{kji}{}^t+(\mathcal{L}_v\eta_t)K_{kji}{}^t=(\mathcal{L}_v\eta_k)g_{ji}-(\mathcal{L}_v\eta_j)g_{ki}+\eta_k\mathcal{L}_v g_{ji}-\eta_j\mathcal{L}_v g_{ki}.$$

Substituting (5.7) into this equation, we obtain

(5.8) $$(\mathcal{L}_v\eta_t)K_{kji}{}^t=-p_k\varphi_{ji}+p_j\varphi_{ki}+2p_i\varphi_{kj}+(\mathcal{L}_v\eta_k)g_{ji}-(\mathcal{L}_v\eta_j)g_{ki} \\ +\eta_k\mathcal{L}_v g_{ji}-\eta_j\mathcal{L}_v g_{ki}.$$

Transvecting (5.8) with η^k and taking account of (5.2), we obtain

(5.9) $$\mathcal{L}_v g_{ji}=p_t\eta^t\varphi_{ji}+\eta_j\eta^k\mathcal{L}_v g_{ki}.$$

Taking account of the symmetric property of $\mathcal{L}_v g_{ji}$ with respect to j and i, we obtain

(5.10) $$2p_t\eta^t\varphi_{ji}+\eta^k(\eta_j\pounds_v g_{ki}-\eta_i\pounds_v g_{kj})=0.$$

Transvecting (5.10) with η^i, we find

(5.11) $$\eta^k\pounds_v g_{kj}=\nu\eta_j,$$

where we have put

(5.12) $$\nu=\eta^k\eta^i\pounds_v g_{ki}.$$

Substituting (5.11) into (5.9), we see that

(5.13) $$\pounds_v g_{ji}=(p_t\eta^t)\varphi_{ji}+\nu\eta_j\eta_i.$$

Transvecting (5.13) with φ^{ji}, we easily find

(5.14) $$p_t\eta^t=0$$

and from which

(5.15) $$\pounds_v g_{ji}=\nu\eta_j\eta_i$$

by virtue of (5.13).

Operating ∇_k to (5.15), we obtain

(5.16) $$\nabla_k(\nabla_j v_i+\nabla_i v_j)=(\nabla_k\nu)\eta_j\eta_i+\nu(\varphi_{kj}\eta_i+\varphi_{ki}\eta_j).$$

Substituting (5.4) into (5.16) and transvecting the result with $\eta^j\eta^i$, we obtain $\nabla_k\nu=0$, that is

(5.17) $$\nu=\text{constant}.$$

On the other hand, substituting (5.14) into the identity:

$$\pounds_v\left\{{h \atop ji}\right\}=\frac{1}{2}g^{ht}(\nabla_j\pounds_v g_{ti}+\nabla_i\pounds_v g_{tj}-\nabla_t\pounds_v g_{ji}),$$

and taking account of (5.17), we obtain

(5.18) $$\pounds_v\left\{{h \atop ji}\right\}=\nu(\varphi_j{}^h\eta_i+\varphi_i{}^h\eta_j).$$

Comparing (5.4) with (5.18), we obtain

(5.19) $$p_j\gamma_i{}^h+p_i\gamma_j{}^h=\nu(\varphi_j{}^h\eta_i+\varphi_i{}^h\eta_j).$$

Transvecting (5.19) with η_i, we easily see that

$$\nu=0$$

by virtue of (5.14), and from which

$$p_i=0,\qquad \pounds_v\left\{{h \atop ji}\right\}=0.$$

Thus we have the following

THEOREM 5.1. *In a Sasakian manifold, an η-projective vector field with an associated vector other than the zero vector does not exist.*

References

1. D.E. Blair, G.D. Ludden and K. Yano, *Hypersurfaces with a parallel vector field in an odd-dimensional sphere,* Differential Geometry in honor of K. Yano, Kinokuniya (1972), 33-39.
2. S.-S. Eum, *On characters of η-related tensors in cosymplectic and Sasakian manifolds* (1), Jour. of Korean Math. Soc. **20** (1983), 173-193.
3. H. Hiramatu, *Riemannian manifold admitting a projective vector field,* Kōdai Math. Jour. **3** (1980), 397-406.
4. M. Obata, *Riemannian manifolds admitting a solution of a certain system of differential equations,* Proc. U.S.-Japan Seminar in Differential Geometry, Kyōto, Japan, 1965, 101-114.
5. K. Yano, *Differential geometry on complex and almost complex spaces,* Pergamon Press, 1965.

Sung Kyun Kwan University
Seoul 110, Korea