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CERTAIN SUBGROUPS OF HOMOTOPY GROUPS

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D.H. Gottlieb ([1], [2]) has defined and studied the evaluation subgroup $G_n(X)$ of the homotopy group $\pi_n(X)$. On the other hand, many authors have studied the homotopy groups of function spaces. In paricular, S. S. Koh ([6]) proved some theorems concerning function spaces. In this paper we will define a subgroup of $\pi_n(X)$ which contains $G_n(X)$. Using the properties of the group defined here, we will generalize the results of S. S. Koh.

1. Introduction

The paper is devided into 5 sections. In section 2 we define a subgroup $G_n(X, A, *)$ of $\pi_n(X, *)$. The relationship between this group and the evaluation map from function space X^A to X is examined and it is shown that $G_n(X, A, *)$ contains $G_n(X, *)$. Moreover we give an example for which $G_n(X, *)<G_n(X, A, *)<\pi_n(X, *)$.

In section 3, we will prove that $G_n(X, A, *)$ is an invariant of homotopy type in the category of spaces homotopically equivalent to CW-pairs.

In section 4, we study some conditions concerning the cell in CW-pair, and some other conditions.

In section 5, we investigate the relationship between G-spaces and W-spaces. In final section 6, we devote ourself to study function spaces and their homotopy groups.

2. Group $G_n(X, A, *)$

Let $(X, *)$ and $(A, *)$ be any two pointed topological spaces and f : $(A, *)\longrightarrow(X, *)$ be a fixed map. Consider the class of continuous functions

$$F : A\times S^n\longrightarrow X$$

such that $F(a, *)=f(a)$.

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Then the map $h:(S^n,*)\longrightarrow(X,*)$ defined by $h(s)=F(*,s)$ represents an element $[h]\in\pi_n(X,*)$.

DEFINITOIN 2.1. The set of all elements $[h]\in\pi_n(X,*)$ obtained in the above manner from some F will be denoted by $G_n^f(X,A,*)$.

Thus for every $[h]\in G_n^f(X,A,*)$, there is at least one map $F:A\times S^n\longrightarrow X$ which satisfies the above conditions. We say that F is an *affiliated map to* $[h]$ *with respect to* A. Note that $[h]$ may have many affiliated maps to $[h]$ with respect to A which are not homotopic. We will abbreviate an affiliated map to $[h]$ with respect to A to an affiliated map to $[h]$ if no confusion arise. It is easy to see that $G_n^f(X,A,*)$ form a subgroup of $\pi_n(X,*)$.

Let A be locally compact and regular, and X^A be the space of mappings from A to X with compact-open-topology. The map $p:X^A\longrightarrow X$ given by $p(g)=g(*)$ is continuous. We call p an evaluation map. Thus p induces homomorphisms

$$p_*:\pi_n(X^A,f)\longrightarrow\pi_n(X,*)$$

for all n. Then we have

THEOREM 2.1. $p_*(\pi_n(X^A,f))=G_n^f(X,A,*)$.

Proof. Since A is locally compact, any continuous map

$$H:(S^n,*)\longrightarrow(X^A,f)$$

gives rise to a continuous associated map

$$\phi(H):A\times S^n\longrightarrow X.$$

Since $\phi(H)(*,s)=(H(s))(*)=(pH)(s)$ and $\phi(H)(a,*)=(H(*))(a)=f(a)$, we have $[pH]=p_*[H]\in G_n^f(X,A,*)$.

Conversely, if F is an affiliated map to $[F(*,)]$, define H by $H=p\phi^{-1}(F)$. Then $[H]\in p_*(\pi_n(X^A,f))$ and $H(s)=F(*,s)$. This completes the theorem.

REMARK. Note that $G_n(X,*,*)=\pi_n(X,*)$ and $G_n^{1_X}(X,X,*)=G_n(X,*)$, where $G_n(X,*)$ is the evaluation subgroup defined by Gottlieb [2].

D. H. Gottlieb proved the following result for CW-complexes A and X [2].

THEOREM 2.2. *For any topological spaces* A,X *and any* $f:(A,*)\longrightarrow(X,*)$, *we have* $G_n(X,*)\leqq G_n^f(X,A,*)$.

THEOREM 2.3. *If A is a subspace of X and $i:(A,*)\longrightarrow(X,*)$ is inclusion, then $G_n^i(X,A,*)\leqq G_n^f(X,A,*)$ for any map $f:(A,*)\longrightarrow(X,*)$ such that $f(A)\subseteq A$.*

Proof. If $[h]\in G_n^i(X,A,*)$, there is an affiliated map

$$F: A\times S^n\longrightarrow X$$

to $[h]$. Define a map $H: A\times S^n\longrightarrow X$ by $H(a,s)=F(f(a),s)$.

DEFINITION 2.2. *$G_n^i(X,A,*)$ will be denoted by $G_n(X,A,*)$.*

DEFINITION 2.3. A space X is an *H-space* iff there a point $*\in X$ and a continuous map $u: X\times X\longrightarrow X$ such that $u(x,*)=u(*,x)=x$ for all $x\in X$. We will write $u(x,y)$ by $x\cdot y$.

THEOREM 2.4. *Suppose that X is an H-space, then*

$$G_n(X,*)=G_n(X,A,*)=\pi_n(X,*).$$

Proof. By Gottlieb ([1], [2]).

The fact that $G_n(X,*)\leqq G_n(X,A,*)\leqq\pi_n(X,*)$ leads naturally to the questions; Is there topological pair (X,A) for which $G_n(X,*)<G_n(X,A,*)<\pi_n(X,*)$? For this we give an example.

EXAMPLE. Let $X=\{z\in C\mid |z|=1,\ |z-2|=1\}$, $A=S^1=\{z\in C\mid |z|=1\}$. Take $*$ by the point $(1,0)$. Define $F: A\times S^1\longrightarrow X$ by $F(z,w)=zw$. Then F is well defined and continuous. Moreover $[F(*,)]$ is one of the generators of the free group $\pi_1(X,*)$ on two generators and $F(*,)$ is inclusion $i:(A,*)\longrightarrow(X,*)$. Thus we have $G_1(X,A,*)=Z$. On the other hand, $G_1(X,*)=\{0\}$ (cf. Gottlieb [1]).

3. Some fundamental theorems

In this section we wish to study the category $\mathscr{C}^2$ whose objects are the pairs (X,A) of spaces and morphisms are maps of pairs.

Let $\sigma: I\longrightarrow X$ be a path such that $\sigma(0)=x_0$ and $\sigma(1)=x_1$. Then σ induces an isomorphism $\sigma_*: G_n(X,x_1)\cong G_n(X,x_0)$. Similarly we will show that, in the usual sense, $G_n(X,A,*)$, viewed as a subgroup of $\pi_n(X,*)$, is independent of the base point.

THEOREM 3.1. *Let (X,A) be a topological pair and $\sigma: I\longrightarrow A\subseteq X$ be a path such that $\sigma(0)=x_0$, $\sigma(1)=x_1$. Then σ induces an isomorphism*

$$\sigma_*: G_n(X,A,x_1)\cong G_n(X,A,x_0).$$

Proof. If $\alpha \in G_n(X, A, x_1)$, there exists an affiliated map

$$F : A \times S^n \longrightarrow X.$$

to α such that $[F(x_1,)] = \alpha$, $F(, *) = i : A \longrightarrow X$. Now define $h : S^n \times I \longrightarrow X$ by

$$h(t, s) = F(\sigma(1-s), t).$$

It is clear that $[h(,0)] = \alpha$ and $h(,1)$ represents $\sigma_*(\alpha) \in \pi_n(X. x_0)$. Moreover F is an affiliated map to $[h(,1)]$. Thus we have $\sigma_*(G_n(X, A, x_1)) \subseteq G_n(X, X, x_0)$.

On the other hand, the reverse path $\sigma^{-1} : I \longrightarrow A \subseteq X$ induces the inverse isomorphism, $(\sigma^{-1})_* : \pi_n(X, x_0) \longrightarrow \pi_n(X, x_1)$, to σ_*. Hence we complete the theorem.

It is not true that $f : (X, A) \longrightarrow (Y, B)$ induces a homomorphism from $G_n(X, A, *)$ to $G_n(Y, B, f(*))$ [1]. But, for some maps, it is true that f_* maps $G_n(X, A, *)$ to $G_n(Y, B, f(*))$. Suppose $r : (Y, B) \longrightarrow (X, A)$ be map. We say that r has a *right homotopy inverse* if there is a map $j : (X, A) \longrightarrow (Y, B)$ such that rj is homotopic to $1_{(X, A)}$ (with homotopy of pair). Similarly we can define a *left homotopy inverse.*

THEOREM 3. 2. *Let (X, A) and (Y, B) be in $\mathscr{T}^2$. Suppose (X, A) is a CW-pair and B is path-connected. If $r : (Y, B) \longrightarrow (X, A)$ has a right homotopy inverse, then $r_* : \pi_n(Y, *) \longrightarrow \pi_n(X, r(*))$ carries $G_n(Y, B, *)$ into $G_n(X, A, r(*))$.*

Proof. First we need a lemma.

LEMMA. *Under the same assumption of Theorem 3. 2, there is a right homotopy inverse*

$$j' : (X, A) \longrightarrow (Y, B) \text{ such that } j'(r(*)) = *.$$

Proof of Lemma. Let $j : (X, A) \longrightarrow (Y, B)$ be a right homotopy inverse of r and $\alpha : I \longrightarrow B \subseteq Y$ be a path such that $\alpha(0) = jr(*)$, $\alpha(1) = *$. By the homotopy extension property, in the diagram

$$\begin{array}{ccc} r(*) \times I \cup A \times 0 & \xrightarrow{\alpha \cup j|A} & B \subseteq Y \\ \downarrow & \nearrow & \\ A \times I & & \end{array}$$

we have an extension $K : A \times I \longrightarrow B \subseteq Y$.

Again in the diagram

$$\begin{array}{ccc} A\times I\cup X\times 0 & \longrightarrow & Y \\ \downarrow & K\cup j \nearrow & \\ X\times I & & \end{array}$$

we have an extension $K' : X\times I\longrightarrow Y$. Let $j'=K'(\ ,1)$, then $j'=K'(\ ,1)\sim K'(\ ,0)=j$ and $rj'\sim rj\sim 1_{(X,A)}$ (homotopy of pair). Moreover $j'(r(*))=K'(r(*),1)=\alpha(1)=*$.

Now we continue the proof of Theorem 3. 2.

If $\alpha\in G_n(Y,B,*)$, there is an affiliated map

$$F : B\times S^n\longrightarrow Y$$

to α. Define $F' : A\times S^n\longrightarrow X$ by

$$F'(a,s)=r(F(j'(a),s)).$$

Then $F'(\ ,*)=rj'$. Since rj' is homotopic to $1_{(X,A)}$, we can find a homotopy $H : (X\times I,\ A\times I)\longrightarrow (X,A)$ such that

$$H|_{A\times 0}=F'|_{A\times *},\ H|_{A\times 1}=1_A.$$

Define a map $G : (A\times *\times I)\cup (A\times S^n\times 0)\longrightarrow X$ by

$$G(a,*,t)=H(a,t),\quad G(a,s,0)=F'(a,s).$$

Then G is well defined and continuous. By the homotopy extension property, we have a homotopy

$$\begin{array}{ccc} (A\times *\times I)\cup (A\times S^n\times 0) & \longrightarrow & X \\ \downarrow & G \nearrow & \\ A\times S^n\times I & & \end{array}$$

$H' : A\times S^n\times I\longrightarrow X$ connecting $H'(\ ,0)=F'$ to $H'(\ ,1)$.

Note that $F'|_{r(*)\times S^n} : S^n\longrightarrow X$ represents $r_*(\alpha)$. Now let $\alpha : I\longrightarrow X$ be given by

$$\sigma(t)=H'(r(*),*,t)\subseteq A.$$

Thus by Theorem 3. 1, σ induces an isomorphism

$$\sigma_* : G_n(X,A,r(*))\cong G_n(X,A,r(*)).$$

Let $h : S^n\longrightarrow X$ be given by $h=H'(r(*),\ ,1)$. Then $\sigma_*[h]=r_*(\alpha)$.

Moreover $[h]\in G_n(X,A,r(*))$. Consequently $r_*(\alpha)=\sigma[h]\in G_n(X,A,r(*))$.

COROLLARY 3. 3. *If* $r:(Y,B)\longrightarrow(X,A)$ *is a retract,* (X,A) *is a* CW*-pair and* B *is path-connected, then* $r_*:\pi_n(Y,*)\longrightarrow\pi_n(X,r(*))$ *carries* $G_n(Y,B,*)$ *into* $G_n(X,A,r(*))$.

THEOREM 3. 4. *Let* (X,A) *and* (Y,B) *be in* $\mathscr{C}^2$. *Let* (X,A) *be a* CW*-pair and* B *be path-connected. If* $j:(Y,B)\longrightarrow(X,A)$ *has a lift homotopy inverse, then* $j_*(\alpha)\in G_n(X,A,x_0)$ *implies* $\alpha\in G_n(Y,B,y_0)$ *where* $j(y_0)=x_0$.

Proof. Since j: $(Y,B)\longrightarrow(X,A)$ has a left homotopy inverse and (X,A) is a CW-pair, we can find $r:(X,A)\longrightarrow(Y,B)$ such that $r(x_0)=y_0$ and $rj\sim 1_{(Y,B)}$ (homotopy of pair) by the homotopy extension property. Let $h_t:Y\longrightarrow Y$ be the homotopy from rj to $1_{(Y,B)}$. Let $\sigma:I\longrightarrow Y$ be a closed path given by $\sigma(t)=h_t(y_0)$. Then $r_*j_*=\sigma_*:\pi_n(Y,y_0)\longrightarrow\pi_n(Y,y_0)$.

If $j_*(\alpha)\in G_n(X,A,x_0)$, then $r_*j_*(\alpha)\in G_n(Y,B,r(x_0))$ by Theorem 3. 2. Hence $\alpha=\sigma_*^{-1}r_*j_*(\alpha)\in G_n(Y,B,y_0)$ by Theorem 3. 1.

Now we can prove that $G_n(X,A)$ is a homotopy type invraiant by using Theorem 3. 2 and 3. 4.

THOREM 3. 5. *Suppose that* (X,A) *and* (Y,B) *are both the same homotopy type of a path-connected* CW*-pair. If* $f:(X,A)\longrightarrow(Y,B)$ *is a homotopy equivalence, then* f_* *carries* $G_n(X,A,*)$ *isomorphically onto* $G_n(Y,B,f(*))$.

Proof. First we assume that (Y,B) is a CW-pair. By Theorem 3. 2, we have $f_*^{-1}(G_n(Y,B,f(*))\subseteq G_n(X,A,*)$. Similarly by Theorem 3. 4, $f_*(G_n(X,A,*))\subseteq G_n(Y,B,f(*))$. Thus $G_n(Y,B)=f_*f_*^{-1}(G_n(Y,B))\subseteq f_*(G_n(X,A))$. Hence $f_*(G_n(X,A))=G_n(Y,B)$. Since f_* is an isomorphsm, the theorem is true for the special case that (Y,B) is a CW-pair.

Now in general, (Y,B) is homotopy equivalent to a CW-pair (Z,C). Let $g:(Y,B)\longrightarrow(Z,C)$ be a homotopy equivalence. Then gf is a homotopy equivalence. Thus g_*f_* carries $G_n(X,A)$ isomorphically onto $G_n(Z,C)$ and g_* carries $G_n(Y,B)$ isomorphically onto $G_n(Z,C)$. Hence f_* must carry $G_n(X,A)$ isomorphically onto $G_n(Y,B)$.

THEOREM 3. 6. *If* (X,A) *and* (Y,B) *are homotopy type of path-connected* CW*-pairs, then*

$$G_n(X\times Y,\ A\times B,\ (x_0,y_0))\cong G_n(X,A,x_0)\oplus G_n(Y,B,y_0).$$

Proof. Since there exists an isomorphism $h : \pi_n(X\times Y, (x_0, y_0)) \longrightarrow \pi_n(X, x_0) \oplus \pi_n(Y, y_0)$ such that $h([\alpha]) = p_*([\alpha]) \oplus q_*([\alpha])$, where p_* and q_* are induced homomorphisms from the projection $X\times Y$ onto X and Y respectively. Now $h(G_n(X\times Y, A\times B, (x_0, y_0)) \subseteq G_n(X, A, x_0) \oplus G_n(Y, B, y_0)$ as may readily be seen by noting that p and q are retractions and applying Corollary 3.3.

On the other hand, let $[\alpha]$ and $[\beta]$ be elements of $G_n(X, A, x_0)$ and $G_n(Y, B, y_0)$ respectively. Now $h^{-1}([\alpha]\oplus[\beta]) = [(j\alpha)\cdot(k\beta)]$ where j and k inject $X \longrightarrow X\times y_0$ and $Y \longrightarrow x_0\times Y$ respectively. Let $H : A\times S^n \longrightarrow X$ be an affiliated map to $[\alpha]$. Define $K : A\times B\times S^n \longrightarrow X\times Y$ such that $K(x, y, s) = (H(x, s), y)$. The existence of K show that $[j\alpha]\in G_n(X\times Y, A\times B, (x_0, y_0))$. Similarly $[k\beta]\in G_n(X\times Y, A\times B, (x_0, y_0))$. Thus the product $[j\alpha]\cdot[k\beta] = [(j\alpha)\cdot(k\beta)] \in G_n(X\times Y, A\times B, (x_0, y_0)$. This completes the Theorem.

4. Relations between $G_n(X\ A, *)$ and $G_n(X, *)$

THEOREM 4.1. *Suppose S is a set of integers, and (X, A) is a CW-pair such that if $e\subset X-A$ is a cell,* dim $e\in S$. *Suppose that if* $m\in S$, $\pi_{n+m-1}(X, *) = \{0\}$. *Then* $G_n(X, A, *) = G_n(X, *)$.

Proof. Since $G_n(X, *) \leqq G_n(X, A, *)$ we need only to prove that $G_n(X, *) \geqq G_n(X, A, *)$. If $[f]\in G_n(X, A, *)$, there is an affiliated map $H : A\times S^n \longrightarrow X$ to $[f]$. Let $L = (A\times S^n)\cup(X\times *)$ and define a map $K : L \longrightarrow X$ by

$$K(a, s) = H(a, s)$$
$$K(x, *) = x.$$

Since $(X\times S^n, L)$ is a CW-pair, there exists an extension K' :

$$\begin{array}{ccc} L & \xrightarrow{\quad K \quad} & X \\ \downarrow & \nearrow_{K'} & \\ X\times S^n & & \end{array}$$

such that $K'|_L = K$. Thus K' is the required affiliated map to $[f]$ with respect to X. Thus we have $[f]\in G_n(X, *)$.

THEOREM 4.2. *Suppose S is a set of integers and (X, A) is a CW-pair such that if $e\subset A-*$ is a cell,* dim$e\in S$. *If* $m\in S$, $\pi_{n+m-1}(X, *) = \{0\}$. *Then* $G_n{}^f(X, A, *) = \pi_n(X, *)$ *for any* $f : A \longrightarrow X$.

Proof. Let $[f]\in\pi_n(X,*)$. Define a map $H:(A\times *)\cup(*\times S^n)\longrightarrow X$ by $H(a,*)=a, H(*,s)=f(s)$. By Corollary 16.3 ([3] p131), there is an extension

$$K: A\times S^n \longrightarrow X$$

such that $K|_{(A\times *)\cup(*\times S^n)}=H$. Thus K is an affiliated map to $[f]$, so that $[f]\in G_n(X,A,*)$.

In particular, if we take $A=X$ in Theorem 4.2, we have

COROLLARY 4.3. *Let X be a CW-complex and $S=\{\dim e|e\in X-*\}$. Suppose that if $m\in S$, then $\pi_{n+m-1}(X,*)=\{0\}$. Then $\pi_n(X,*)=G_n(X,*)=G_n{}^f(X,A,*)$ for any $A\subset X$ and for any $f: A\longrightarrow X$.*

In general, $i_*(\pi_n(A,*))\not\subseteq G_n(X,A,*)$ for $i: A\longrightarrow X$. But under the same assumption of Theorem 4.2, we have $i_*(\pi_n(A,*))\subseteq G_n(X,A,*)$.

THEOREM 4.4. *If A is a retract of X, then*
$$G_n(X,A,*)\cap i_*(\pi_n(A,*))=i_*(G_n(A,*)).$$

Proof. $G_n(X,A,*)\cap i_*(\pi_n(A,*))\geqq i_*(G_n(A,*))$ is obvious. Conversely, if $[f]\in G_n(X,A,*)\cap i_*(\pi_n(A,*))$, there is a map $g:(S^n,*)\longrightarrow(A,*)$ such that $i_*[g]=[ig]=[f]$. And there is an affiliated map

$$F: A\times S^n\longrightarrow X$$

to $[f]$. Define $F': A\times S^n\longrightarrow A$ by $F'=rF$, where $r: X\longrightarrow A$ is a retraction. Then $[F'(*,\ )]=[rf]=r_*[f]=[rig]=[g]$. And $F'(\ ,*)=ri=1_A$. This implies $[f]=i_*[g]\in i_*(G_n(A,*))$.

COROLLARY 4.5. *If A is a retract of X, we have*
$$G_1(X,A,*)\ \cap i_*(\pi_1(A,*))\subseteq i_*(Z(\pi_1(A,*))$$
where $Z(A)$ denotes the center of the group A.

Proof. Gottlieb [1].

COROLLARY 4.6. *Let A be a compact polyhedron such that Euler-Poincare number $\chi(A)\neq 0$ and be a retract of X. Then we have*

$$G_1(X,A,*)\cap i_*(\pi_1(A,*))=\{0\}.$$

Proof. Gottlieb [1].

THEOREM 4.7. *Let X be a CW-complex and $\{X_\alpha\}$ be the set of all finite subcomplexes. Then $G_n(X,x)=\varprojlim G_n(X,X_\alpha,*)$.*

Proof. Since $\{X_\alpha\}$ directed by indusion ($X_\alpha \leqq X_\beta$ iff $X_\alpha \subseteq X_\beta$), we can construct an inverse system

$$\{G_n(X, X_\alpha, *), (i_{X_\alpha X_\beta})_*, \{X_\alpha\}\}$$

where $(i_{X_\alpha X_\beta})_* : G_n(X, X_\beta, *) \leqq G_n(X, X_\alpha, *)$ is an inclusion homomorphism. By the defintion of inverse limit of the inverse system, we obtain the required result.

5. *G*-spaces and *W*-spaces

Define a subgroup $P_n(X, A, *)$ of $\pi_n(X, *)$ as follows:

DEFINITION 5.1. $[f] \in P_n(X, A, *)$ iff for every $[g] \in \pi_m(A, *)$ and every m, there exists a map $G : S^m \times S^n \longrightarrow X$ such that $G(*,) = f, G(, *) = ig$, where $i : (A, *) \longrightarrow (X, *)$ is an inclusion.

Then $P_n(X, X, *)$ is the group $P_n(X, *)$ defined by Gottlieb [2], that is $P_n(X, *) = \{[f] \in \pi_n(X, *) \mid [[f], [g]] = 0$ (Whitehead product) for every m and every element $[g] \in \pi_m(X, *)\}$.

THEOREM 5.1. $G_n(X, A, *) \leqq P_n(X, A, *)$.

Proof. Let $[\alpha] \in G_n(X, A, *)$. Then there is an affiliated map $F : A \times S^n \longrightarrow X$ to $[\alpha]$. Let $[g] \in \pi_m(A, *)$. Define a map $G : S^m \times S^n \longrightarrow X$ given by $G(r, s) = F(g(r), s)$. Then $G(*,) = F(*,) = \alpha$, $G(, *) = F(g( ), *) = ig$. Thus $[\alpha] \in P_n(X, A, *)$.

THEOREM 5.2. $P_n(X, *) \leqq P_n(X, A, *)$.

Proof. It is obvious.

DEFINITION 5.2. (a) A *G*-space is a space X with $G_n(X, *) = \pi_n(X, *)$ for all n.

(b) A *W*-space is a space X with $P_n(X, *) = \pi_n(X, *)$ for all n.

COROLLARY 5.3. *If X is G-space, then*

$$G_n(X, *) = G_n(X, A, *) = P_n(X, *) = P_n(X, A, *) = \pi_n(X, *).$$

COROLLARY 5.4. *If X is a W-space, then*

$$P_n(X, *) = P_n(X, A, *) = \pi_n(X, *).$$

THEOREM 5.5. *If $f : (S^m, *) \longrightarrow (X, *)$ is any continuous map, then we have $P_n(X, *) \leqq G_n^f(X, S^m, *)$.*

Proof. Let $\alpha \in P_n(X, *)$ with $\alpha = [g]$. Since the given $f : (S^m, *) \longrightarrow$

$(X,*)$ is continuous, $[f]\in\pi_m(X,*)$. Thus there is a map $F: S^m\times S^n \longrightarrow X$ such that $F(*,)=g$ and $F(,*)=f$. The existence of F implies $\alpha\in G_n{}^f(X, S^m, *)$.

COROLLARY 5.6. $P_n(X,*)=\bigcap_{f,m} G_n{}^f(X,S^m,*)$

COROLLARY 5.7. [7]. $P_n(S^n,*)=G_n(S^n,*)$

$$=\begin{cases}0 & \textit{for } n \textit{ even}\\ Z & n=1,\ 3 \textit{ or } 7\\ 2Z & n=\textit{odd},\ n\neq 1, 3 \textit{ or } 7.\end{cases}$$

6. Function space X^A

Now suppose that A is a locally compact and ANR space, then the evaluation map $p: X^A \longrightarrow X$ is a fibring ([5]).

Let F be the fibre $p^{-1}(*)$, then we have a long exact sequence $\cdots\longrightarrow\pi_n(F,\ f)\longrightarrow\pi_n(X^A,\ f)\longrightarrow\pi_n(X,*)\longrightarrow\pi_{n-1}(F,\ f)\longrightarrow\cdots$ where $f: (A,*)\longrightarrow(X,*)$.

By Theorem 2.1, we have the following theorem.

THEOREM 6.1. *The next three statements are equivalent:*

(i) p_* *is epimorphism*

(ii) $G_n{}^f(X,A,*)=\pi_n(X,*)$

(iii) *For any* $g: (S^n,*)\longrightarrow(X,*)$, *there is a lift* $\tilde{g}: (S^n,*)\longrightarrow(X^A, f)$ *such that* $[p\tilde{g}]=[g]$.

Combining theorem 6.1 and proposition 6.2([5]p. 152) we have

COROLLARY 6.2. *If the fibering* $p: X^A\longrightarrow X$ *admits a cross section* $\alpha: X\longrightarrow X^A$, *then*

$$G_n{}^f(X,A,*)=\pi_n(X,*)\quad(n\geqslant 1).$$

THEOREM 6.3. *If* X *is an* H-*space with* $*$ *as unit, then we have* $\pi_n(X^A,f)\cong\pi_n(F,f)\oplus\pi_n(X,*)$ $(n\geqslant 1)$.

Proof. Define a map $\alpha: X\longrightarrow X^A$ by

$$(\alpha(x))(a)=x\cdot f(a).$$

Then α is well defined because $\alpha(x): A\longrightarrow X$ is continuous. Moreover $p\alpha=1_X$.

Now we will show that α is continuous: Since A is locally compact, $\alpha : X \longrightarrow X^A$ is continuous if and only if $\alpha : X \times A \longrightarrow X$ is continuous. But the continuity of α is clear.

Moreover $\alpha(*) = f$. This α is a cross-section. Thus we have the results for $n \geqslant 2$ by proposition 6. 2([5]p. 152).

In case $n=1$, we have the short exact sequence, since X is an H-space,

$$0 \longrightarrow \pi_1(F, f) \longrightarrow \pi_1(X^A, f) \longrightarrow \pi_1(X, *) \longrightarrow 0.$$

Since $\alpha : X \longrightarrow X^A$ is a cross-section, it induces a homomorphism $\alpha_* : \pi_1(X, *) \longrightarrow \pi_1(X^A, f)$ and $p_* \alpha_* = (p\alpha)_* = 1$. Thus need only to prove $\pi_1(X^A, f)$ is abelian. But the two multiplications in $[(SA, *), (X, *)]$ are the same and they are commutative ([3] p. 65). So that $[S, X^A] \cong \pi_1(X^A)$ is abelian. This completes the theorem.

COROLLARY 6. 4. *Let X be an H-space. Then*

$$\pi_n(X^{s^q}, f) \cong \pi_n(X, *) \oplus \pi_{n+q}(X, *) \quad n \geqslant 1$$

Proof. By Whitehead theorem, we have $\pi_n(F, f) \cong \pi_{n+q}(X, *)$.

COROLLARY 6. 5.
$$\begin{cases} \pi_1(({S^1}_\times S^1)^{S^0}) \cong Z \oplus Z \oplus Z \oplus Z \\ \pi_1(({S^1}_\times S^1)^{S^q}) \cong Z \oplus Z \quad (q>0) \\ \pi_n(({S^1}_\times S^1)^{S^q}) \cong \{0\} \quad (n>1, \ q \geqslant 0). \end{cases}$$

COROLLARY 6. 6. *Let π be abelian and $K(\pi, n)$ be an Eilenberg-Maclane space. Then*

$$\pi_m(K(\pi, n)^{S^q}) \cong \begin{cases} \left.\begin{matrix} \pi \oplus \pi & m=n \\ \{0\} & m \neq n \end{matrix}\right\} & (q=0) \\ \left.\begin{matrix} \pi & m=n \\ \pi & m+q=n \\ 0 & otherxise \end{matrix}\right\} & (q>0) \end{cases}$$

Proof. Since π is abelian, $K(\pi, n)$ is an H-space.

REMARK. (1). S. S. Koh [6] proved that if X is an H-space then $\pi_p({}_X S^q) / \pi_{p+q}(X) \cong \pi_p(X)$. We generalized this.

(2). We can calculate the homotopy groups for $S^1 \times S^1 \times \cdots \times S^1$, $S^3 \times S^3 \times \cdots \times S^3$ and so on.

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