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ON A SPECIAL CLASS OF UNIVALENT FUNCTIONS IN THE UNIT DISK

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1. Introduction

Throughout this paper, let

$$f(z)=a_1z-\sum_{n=2}^{\infty}a_nz^n \qquad (a_1>0, a_n\geqq 0),$$

$$f_i(z)=a_{1,i}z-\sum_{n=2}^{\infty}a_{n,i}z^n \qquad (a_{1,i}>0, a_{n,i}\geqq 0)$$

and

$$g(z)=b_1z-\sum_{n=2}^{\infty}b_nz^n \qquad (b_1>0, \ b_n\geqq 0)$$

and let $\mathcal{D}^*(\alpha,\beta,\gamma)$ denote a class of functions $f(z)$ analytic and univalent in the unit disk $\mathcal{U}=\{z: |z|<1\}$ for which

$$\left|\frac{f'(z)-1}{2\gamma\{f'(z)-\alpha\}-\{f'(z)-1\}}\right|<\beta,$$

where $0\leqq\alpha<1, 0<\beta\leqq 1$ and $0<\gamma\leqq 1$. The above condition on $f'(z)$ implies the univalence of $f(z)$.

For this class $\mathcal{D}^*(\alpha,\beta,\gamma)$, V.P. Gupta and I. Ahmad [1] showed the following lemma.

LEMMA 1. *A function $f(z)$ is in the class $\mathcal{D}^*(\alpha,\beta,\gamma)$ if, and only if,*

$$\sum_{n=2}^{\infty}na_n\leqq\frac{2\beta\gamma(1-\alpha)a_1}{1+2\beta\gamma-\beta}.$$

REMARK 1. V.P. Gupta and P.K. Jain [2] have studied a class $\mathcal{D}^*$ (α,β) of functions

$$f(z)=z-\sum_{n=2}^{\infty}a_nz^n \qquad (a_n\geqq 0)$$

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analytic and univalent in the unit disk $\mathcal{U}$ for which

$$\left|\frac{f'(z)-1}{f'(z)+(1-2\alpha)}\right|<\beta,$$

where $0\leqq\alpha<1$ and $0<\beta\leqq1$. If we put $\gamma=1$ and $a_1=1$, then

$$\mathcal{P}^*(\alpha,\beta,1)=\mathcal{P}^*(\alpha,\beta).$$

REMARK 2. Let $0\leqq\alpha_1\leqq\alpha_2<1$. Then we can see that

$$\mathcal{P}^*(\alpha_2,\beta,\gamma)\subset\mathcal{P}^*(\alpha_1,\beta,\gamma)$$

with the aid of Lemma 1.

REMARK 3. Let $0\leqq\alpha_1\leqq\alpha_2<1$, $0<\beta_1\leqq\beta_2\leqq1$ and $0<\gamma\leqq1/2$. Then we can see that

$$\mathcal{P}^*(\alpha_2,\beta_1,\gamma)\subset\mathcal{P}^*(\alpha_1,\beta_2,\gamma)$$

with Lemma 1 and the definition of $\mathcal{P}^*(\alpha,\beta,\gamma)$.

2. The Hadamard products of the functions in $\mathcal{P}^*(\alpha,\beta,\gamma)$

Let $f*g(z)$ denote the Hadamard product of two functions $f(z)$ and $g(z)$, that is,

$$f*g(z)=a_1b_1z+\sum_{n=2}^{\infty}a_nb_nz^n.$$

THEOREM 1. *Let the functions $f_i(z)$ be in the classes $\mathcal{P}^*(\alpha_i,\beta,\gamma)$ for each $i=1,2,\dots,m$, respectively. Then, the Hadamard product $f_1*f_2*\dots*f_m(z)$ is in the class $\mathcal{P}^*(\alpha,\beta,\gamma)$, where*

$$\alpha=1-2^{1-m}\prod_{i=1}^{m}(1-\alpha_i).$$

Proof. Since $f_i(z)\in\mathcal{P}^*(\alpha_i,\beta,\gamma)$ for each $i=1,2,\dots,m$, respectively, by using Lemma 1,

$$\sum_{n=2}^{\infty}n(1+2\beta\gamma-\beta)a_{n,i}\leqq2\beta\gamma(1-\alpha_i)a_{1,i}$$

and

$$a_{n,i}\leqq\frac{\beta\gamma(1-\alpha_i)a_{1,i}}{1+2\beta\gamma-\beta}$$

for any $n\geqq2$. Hence, we have

$$\begin{aligned}
&\sum_{n=2}^{\infty} n(1+2\beta\gamma-\beta)\prod_{i=1}^{m} a_{n,i}\\
&\leqq \frac{\beta^{m-1}\gamma^{m-1}}{(1+2\beta\gamma-\beta)^{m-1}}\prod_{i=1}^{m-1}(1-\alpha_i)a_{1,i}\sum_{n=2}^{\infty} n(1+2\beta\gamma-\beta)a_{n,m}\\
&\leqq \frac{2\beta^m\gamma^m}{(1+2\beta\gamma-\beta)^{m-1}}\prod_{i=1}^{m}(1-\alpha_i)a_{1,i}\\
&\leqq 2\beta\gamma\left\{1-\left(1-2^{1-m}\prod_{i=1}^{m}(1-\alpha_i)\right)\right\}\prod_{i=1}^{m}a_{1,i}.
\end{aligned}$$

This completes the proof of the theorem with the aid of Lemma 1.

COROLLARY 1. *A class $\mathcal{D}^*(\alpha,\beta,\gamma)$ is closed under the Hadamard product.*

Proof. Let the functions $f(z)$ and $g(z)$ be in the same class $\mathcal{D}^*(\alpha,\beta,\gamma)$. Then, the Hadamard product $f*g(z)$ belongs to the class $\mathcal{D}^*\{(1+2\alpha-\alpha^2)/2,\beta,\gamma\}$ by means of Theorem 3. Furthermore, since $(1+2\alpha-\alpha^2)/2$ is greater than α, by using Theorem 1, we have $f*g(z)\in\mathcal{D}^*(\alpha,\beta,\gamma)$.

THEOREM 2. *Let the functions $f_i(z)$ be in the classes $\mathcal{D}^*(\alpha_i,\beta,\gamma)$ for each $i=1,2,\dots,m$, respectively. Then, the function*

$$h(z)=\sum_{i=1}^{m} f_i(z)$$

is in the class $\mathcal{D}^*(\alpha,\beta,\gamma)$, where $\alpha=\min\limits_{1\leqq i\leqq m}\alpha_i$.

Proof. From the definition of $h(z)$, we have

$$h(z)=\left(\sum_{i=1}^{m}a_{1,i}\right)z-\sum_{n=2}^{\infty}\left(\sum_{i=1}^{m}a_{n,i}\right)z^n.$$

And since $f_i(z)\in\mathcal{D}^*(\alpha_i,\beta,\gamma)$ for each $i=1,2,\dots,m$, respectively, by Lemma 1,

$$\sum_{n=2}^{\infty} n(1+2\beta\gamma-\beta)a_{n,i}\leqq 2\beta\gamma(1-\alpha_i)a_{1,i}.$$

Hence,

$$\begin{aligned}
&\sum_{n=2}^{\infty} n(1+2\beta\gamma-\beta)\sum_{i=1}^{m}a_{n,i}\\
&\leqq 2\beta\gamma\sum_{i=1}^{m}(1-\alpha_i)a_{1,i}
\end{aligned}$$

$$\leqq 2\beta\gamma(1-\min_{1\leqq i\leqq m}\alpha_i)\sum_{i=1}^{m}a_{1,i}.$$

Therefore, we have the theorem with Lemma 1.

3. Distortion theorems for the fractional calculus

THEOREM 3. *Let a function $f(z)$ be in the class $\mathcal{P}^*(\alpha,\beta,\gamma)$. Then, we have*

$$|f(z)|\geqq a_1|z|-\frac{\beta\gamma(1-\alpha)a_1}{1+2\beta\gamma-\beta}|z|^2,$$

$$|f(z)|\leqq a_1|z|+\frac{\beta\gamma(1-\alpha)a_1}{1+2\beta\gamma-\beta}|z|^2,$$

$$|f'(z)|\geqq a_1-\frac{2\beta\gamma(1-\alpha)a_1}{1+2\beta\gamma-\beta}|z|$$

and

$$|f'(z)|\leqq a_1+\frac{2\beta\gamma(1-\alpha)a_1}{1+2\beta\gamma-\beta}|z|$$

for $z\in\mathcal{U}$.

Proof. Since $f(z)\in\mathcal{P}^*(\alpha,\beta,\gamma)$, by using Lemma 1,

$$\sum_{n=2}^{\infty}a_n\leqq\frac{\beta\gamma(1-\alpha)a_1}{1+2\beta\gamma-\beta}$$

and

$$\sum_{n=2}^{\infty}na_n\leqq\frac{2\beta\gamma(1-\alpha)a_1}{1+2\beta\gamma-\beta}.$$

Hence, we have

$$|f(z)|\geqq a_1|z|-|z|^2\sum_{n=2}^{\infty}a_n\geqq a_1|z|-\frac{\beta\gamma(1-\alpha)a_1}{1+2\beta\gamma-\beta}|z|^2,$$

$$|f(z)|\leqq a_1|z|+|z|^2\sum_{n=2}^{\infty}a_n\leqq a_1|z|+\frac{\beta\gamma(1-\alpha)\alpha_1}{1+2\beta\gamma-\beta}|z|^2,$$

$$|f'(z)|\geqq a_1-|z|\sum_{n=2}^{\infty}na_n\geqq a_1-\frac{2\beta\gamma(1-\alpha)a_1}{1+2\beta\gamma-\beta}|z|$$

and

$$|f'(z)| \leqq a_1 + |z| \sum_{n=2}^{\infty} n a_n \leqq a_1 + \frac{2\beta\gamma(1-\alpha)a_1}{1+2\beta\gamma-\beta}|z|.$$

Finally, the equalities hold for the function

$$f(z) = a_1 z - \frac{\beta\gamma(1-\alpha)a_1}{1+2\beta\gamma-\beta} z^2.$$

COROLLARY 2. *Under the hypothesis of Theorem 3, the unit disk $\mathcal{U}$ is mapped onto a domain that contains the disk*

$$|w| < a_1 - \frac{\beta\gamma(1-\alpha)a_1}{1+2\beta\gamma-\beta}.$$

This result is sharp with an extremal function

$$f(z) = a_1 z - \frac{\beta\gamma(1-\alpha)a_1}{1+2\beta\gamma-\beta} z^2.$$

Next, let $D_z^{-k}f(z)$ and $D_z^k f(z)$ denote the fractional integral of order k and the fractional derivative of order k, respectively. In 1978, S. Owa [5] gave the following definitions for the fractional calculus.

DEFINITION 1. The fractional integral of order k is defined by

$$D_z^{-k}f(z) = \frac{1}{\Gamma(k)} \int_0^z \frac{f(\zeta)d\zeta}{(z-\zeta)^{1-k}},$$

where $k>0$, $f(z)$ is an analytic function in a simply connected region of the z-plane containing the origin and the multiplicity of $(z-\zeta)^{k-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta)>0$.

DEFINITION 2. The fractional derivative of order k is defined by

$$D_z^k f(z) = \frac{1}{\Gamma(1-k)} \frac{d}{dz} \int_0^z \frac{f(\zeta)d\zeta}{(z-\zeta)^k},$$

where $0<k<1$, $f(z)$ is an analytic function in a simply connected region of the z-plane containing the origin and the multiplicity of $(z-\zeta)^{-k}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta)>0$.

REMARK 4. The fractional derivative of order $(n+k)$ is defined by

$$D_z^{n+k}f(z)=\frac{d^n}{dz^n}D_z^k f(z),$$

where $0<k<1$ and $n\in\mathcal{N}\cup\{0\}$.

REMARK 5. For other definitions of the fractional calculus, see [3], [4], [6] and [7].

THEOREM 4. *Let a function $f(z)$ be in the class $\mathcal{P}^*(\alpha,\beta,\gamma)$. Then, we have*

$$|D_z^{-k}f(z)|\geqq\frac{a_1}{\Gamma(2+k)}|z|^{1+k}-\frac{\beta\gamma(1-\alpha)a_1}{(1+2\beta\gamma-\beta)\Gamma(2+k)}|z|^{2+k},$$

$$|D_z^{-k}f(z)|\leqq\frac{a_1}{\Gamma(2+k)}|z|^{1+k}+\frac{\beta\gamma(1-\alpha)a_1}{(1+2\beta\gamma-\beta)\Gamma(2+k)}|z|^{2+k},$$

$$|D_z^{1-k}f(z)|\geqq\frac{(1-k)a_1}{\Gamma(2+k)}|z|^{k}-\frac{\beta\gamma(2+k)(1-\alpha)a_1}{(1+2\beta\gamma-\beta)\Gamma(2+k)}|z|^{1+k}$$

and

$$|D_z^{1-k}f(z)|\leqq\frac{(1+k)a_1}{\Gamma(2+k)}|z|^{k}+\frac{\beta\gamma(2+k)(1-\alpha)a_1}{(1+2\beta\gamma-\beta)\Gamma(2+k)}|z|^{1+k}$$

for $0<k<1$ and $z\in\mathcal{U}$.

Proof. Let

$$\begin{aligned}F(z)&=\Gamma(2+k)z^{-k}D_z^{-k}f(z)\\&=a_1z-\sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(2+k)}{\Gamma(n+1+k)}a_nz^n\\&=a_1z-\sum_{n=2}^{\infty}A_zz^n.\end{aligned}$$

Then, by using Lemma 1, we have

$$\sum_{n=2}^{\infty}n(1+2\beta\gamma-\beta)A_n<\sum_{n=2}^{\infty}n(1+2\beta\gamma-\beta)a_n\leqq 2\beta\gamma(1-\alpha)a_1,$$

because $0<A_n<a_n$. Hence, the function $F(z)$ belongs to the class $\mathcal{P}^*(\alpha,\beta,\gamma)$. From this, we have the required estimates with the aid of Theorem 3.

THEOREM 5. *Let a function $f(z)$ be in the class $\mathcal{P}^*(\alpha,\beta,\gamma)$. Then, we have*

$$|D_z^k f(z)| \geqq \frac{ka_1}{\Gamma(3-k)}|z|^{1-k} - \frac{\beta\gamma(3-k)(1-\alpha)a_1}{(1+2\beta\gamma-\beta)\Gamma(3-k)}|z|^{2-k}$$

and

$$|D_z^k f(z)| \leqq \frac{(2-k)a_1}{\Gamma(3-k)}|z|^{1-k} + \frac{\beta\gamma(3-k)(1-\alpha)a_1}{(1+2\beta\gamma-\beta)\Gamma(3-k)}|z|^{2-k}$$

for $0<k<1$ *and* $z\in\mathcal{U}$.

Proof. Let consider the function

$$\begin{aligned} G(z) &= \Gamma(3-k)z^{-1+k}D_z^{-1+k}f(z) \\ &= a_1 z - \sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(3-k)}{\Gamma(n+2-k)}a_n z^n \\ &= a_1 z - \sum_{n=2}^{\infty} B_n z^n. \end{aligned}$$

Then, by Lemma 1, we have $G(z)\in\mathcal{P}^*(\alpha,\beta,\gamma)$, because $0<B_n<a_n$. Hence,

$$|D_z^{-1+k} f(z)| \leqq \frac{a_1}{\Gamma(3-k)}|z|^{2-k} + \frac{\beta\gamma(1-\alpha)a_1}{(1+2\beta\gamma-\beta)\Gamma(3-k)}|z|^{3-k}$$

by using the second estimate of Theorem 3. Furthermore, from this and the third estimate and the fourth estimate of Theorem 3, we get the required estimates.

COROLLARY 3. *Under the hypothesis of Theorem* 5, *the unit disk* $\mathcal{U}$ *is mapped onto a domain that contains the disk*

$$|w| < \frac{ka_1}{\Gamma(3-k)} - \frac{\beta\gamma(3-k)(1-\alpha)a_1}{(1+2\beta\gamma-\beta)\Gamma(3-k)}.$$

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