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AN APPLICATION OF THE FRACTIONAL CALCULUS II

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1. Introduction

Let $\mathcal{A}_p$ denote the class of functions of the form

$$f(z)=z^p+\sum_{n=1}^{\infty}a_{p+n}z^{p+n} \quad (p\in \mathcal{N}=\{1, 2, 3, \dots\}) \tag{1.1}$$

which are analytic in the unit disk $\mathcal{U}=\{z : |z|<1\}$. Then a function $f(z)\in\mathcal{A}_p$ is said to be in the class $\mathcal{S}_p{}^*(\alpha)$ if

$$\mathrm{Re}\left\{\frac{zf'(z)}{pf(z)}\right\}>\alpha \qquad (z\in U) \tag{1.2}$$

for $0\leqq\alpha<1$. In particular, $\mathcal{S}_1{}^*(\alpha)$ is the class of analytic and starlike functions of order α.

Now, we say that a function $f(z)$ defined by (1.1) is in the class $\mathcal{V}_p(\theta_{p+n})$ if $f(z)\in\mathcal{A}_p$ and $\arg\{a_{p+n}\}=\theta_{p+n}$ for all $n\geqq1$. Then it is clear that $\mathcal{V}_p(\theta_{p+n}+2k\pi)=\mathcal{V}_p(\theta_{p+n})$ for an integer k. If $a_{p+m}=0$ for some m, we are then to choose θ_{p+m} arbitrarily. Thus even if we were to restrict $\arg\{a_{p+n}\}$ to principal values, $f(z)$ would not necessarily be in a unique $\mathcal{V}_p(\theta_{p+n})$. If, further, there exists a real number β such that

$$\theta_{p+n}+n\beta\equiv\pi \pmod{2\pi} \tag{1.3}$$

then $f(z)$ is said to be in the class $\mathcal{V}_p(\theta_{p+n};\beta)$. The union of $\mathcal{V}_p(\theta_{p+n};\beta)$ taken over all possible sequences $\{\theta_{p+n}\}$ and all possible real number β is denote by $\mathcal{V}_p$. Further let $\mathcal{V}_p{}^*(\alpha)$ denote the subclass of $\mathcal{V}_p$ consisting of functions $f(z)$ in $\mathcal{S}_p{}^*(\alpha)$.

REMARK 1. The classes $\mathcal{V}_1$ and $\mathcal{V}_1{}^*(\alpha)$ were first studied by Silverman [10]. Further Owa [6] showed some distortion theorems for fractional calculus of functions $f(z)$ in $\mathcal{V}_1{}^*(\alpha)$.

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There are many definitions of the fractional calculus, that is, the fractional integrals and the fractional derivatives. We find it convenient to restrict ourselves to the following definitions of the fractional calculus used recently by Owa [7].

DEFINITION 1. The fractional integral of order λ is defined by

$$D_z^{-\lambda}f(z)=\frac{1}{\Gamma(\lambda)}\int_0^z \frac{f(\zeta)d\zeta}{(z-\zeta)^{1-\lambda}}, \tag{1.4}$$

where $\lambda>0$, $f(z)$ is an analytic function in a simply connected region of the z-plane containing the origin and the multiplicity of $(z-\zeta)^{\lambda-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta)>0$.

DEFINITION 2. The fractional derivative of order λ is defined by

$$D_z^{\lambda}(z)=\frac{1}{\Gamma(1-\lambda)}\frac{d}{dz}\int_0^z \frac{f(\zeta)d\zeta}{(z-\zeta)^{\lambda}}, \tag{1.5}$$

where $0<\lambda<1$, $f(z)$ is an analytic function in a simply connected region of the z-plane containing the origin and the multiplicity of $(z-\zeta)^{-\lambda}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta)>0$.

DEFINITION 3. Under the hypotheses of Definition 2, the fractional derivative of order $(n+\lambda)$ is defined by

$$D_z^{n+\lambda}f(z)=\frac{d^n}{dz^n}D_z^{\lambda}f(z), \tag{1.6}$$

where $0<\lambda<1$ and $n\in\mathcal{N}\cup\{0\}$.

REMARK 2. For other definitions of the fractional calculus, see Agarwal [1], Al-Salam [2], Erdélyi, Magnus, Oberhettinger and Tricomi [3], Nishimoto [4], Osler [5], Ross [8], Saigo [9]. Sneddon [11] and Srivastava and Buschman [12].

2. Distortion Theorems

In this section, we show a coefficient inequality and two distortion theorems of functions $f(z)$ in $\mathcal{P}_p^*(\alpha)$.

THEOREM 1. *Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{P}_p^*(\alpha)$. Then we have*

$$\sum_{n=1}^{\infty}\left\{\left(\frac{p+n}{p}\right)-\alpha\right\}|a_{p+n}|\leqq 1-\alpha. \tag{2.1}$$

Proof. We use a method of Silverman [10]. We assume that

$$\text{(2.2)}\qquad \operatorname{Re}\left\{\frac{zf'(z)}{pf(z)}\right\}=\operatorname{Re}\left\{\frac{1+\sum_{n=1}^{\infty}\frac{p+n}{p}a_{p+n}z^n}{1+\sum_{n=1}^{\infty}a_{p+n}z^n}\right\}>\alpha$$

for $0\leqq\alpha<1$ and $z\in\mathcal{U}$. For $f(z)\in\mathcal{V}_p(\theta_{p+n};\beta)$ we put $z=re^{i\beta}$ in (2.2) and let $r\to1^-$. Upon clearing the denominator in (2.2) we can see that

$$\text{(2.3)}\qquad 1-\sum_{n=1}^{\infty}\left(\frac{p+n}{p}\right)|a_{p+n}|\geqq\alpha\left(1-\sum_{n=1}^{\infty}|a_{p+n}|\right)$$

which shows (2.1). Thus we have the theorem.

THEOREM 2. *Let the function $f(z)$ defined by* (1.1) *be in the class $\mathcal{V}_p^*(\alpha)$. Then we have*

$$\text{(2.4)}\qquad |z|^p-\frac{p(1-\alpha)}{p(1-\alpha)+1}|z|^{p+1}\leqq|f(z)|\leqq|z|^p+\frac{p(1-\alpha)}{p(1-\alpha)+1}|z|^{p+1}$$

for $z\in\mathcal{U}$. Equality occurs for

$$\text{(2.5)}\qquad f(z)=z^p+\frac{p(1-\alpha)}{p(1-\alpha)+1}z^{p+1}$$

at $z=\pm|z|\exp(-i\theta_{p+1})$.

Proof. Since $f(z)\in V_p^*(\alpha)$, in view of Theorem 1, we can see that

$$\text{(2.6)}\qquad \sum_{n=1}^{\infty}|a_{p+n}|\leqq\frac{p(1-\alpha)}{p(1-\alpha)+1}$$

Hence we obtain

$$\text{(2.7)}\qquad |f(z)|\leqq|z|^p+\sum_{n=1}^{\infty}|a_{p+n}||z|^{p+n}\leqq|z|^p+|z|^{p+1}\sum_{n=1}^{\infty}|a_{p+n}|\leqq|z|^p+\frac{p(1-\alpha)}{p(1-\alpha)+1}|z|^{p+1}$$

Similarly, we have

$$\text{(2.8)}\qquad |f(z)|\geqq|z|^p-\sum_{n=1}^{\infty}|a_{p+n}||z|^{p+n}$$

$$\geqq |z|^p - |z|^{p+1}\sum_{n=1}^{\infty}|a_{p+n}|$$
$$\geqq |z|^p - \frac{p(1-\alpha)}{p(1-\alpha)+1}|z|^{p+1}.$$

This completes the proof of the theorem.

COROLLARY 1. *Under the hypotheses of* Theorem 2, *$f(z)$ is included in a disk with its center at the origin and radius r given by*

$$r = 1 + \frac{p(1-\alpha)}{p(1-\alpha)+1}. \tag{2.9}$$

THEOREM 3. *Let the function $f(z)$ defined by* (1.1) *be in the class $\mathcal{D}_p^*(\alpha)$. Then we have*

$$p|z|^{p-1} - \frac{p(p+1)(1-\alpha)}{p(1-\alpha)+1}|z|^p \leqq |f'(z)| \tag{2.10}$$
$$\leqq p|z|^{p-1} + \frac{p(p+1)(1-\alpha)}{p(1-\alpha)+1}|z|^p$$

for $0 \leqq \alpha < 1$ and $z \in \mathcal{U}$. Equality occurs for the function $f(z)$ is given by (2.5).

Proof. By using $f(z) \in \mathcal{D}_p^*(\alpha)$ and

$$\left(\frac{p(1-\alpha)+1}{p(p+1)}\right)(p+n) \leqq \left(\frac{p+n}{p}\right) - \alpha \tag{2.11}$$

for $0 \leqq \alpha < 1$ and $n \geqq 1$, we get

$$\left(\frac{p(1-\alpha)+1}{p(p+1)}\right)\sum_{n=1}^{\infty}(p+n)|a_{p+n}| \tag{2.12}$$
$$\leqq \sum_{n=1}^{\infty}\left\{\left(\frac{p+n}{p}\right)-\alpha\right\}|a_{p+n}|$$
$$\leqq 1-\alpha,$$

hence further,

$$\sum_{n=1}^{\infty}(p+n)|a_{p+n}| \leqq \frac{p(p+1)(1-\alpha)}{p(1-\alpha)+1}. \tag{2.13}$$

Consequently we can show that

$$|f'(z)| \leqq p|z|^{p-1} + \sum_{n=1}^{\infty}(p+n)|a_{p+n}||z|^{p+n-1} \tag{2.14}$$
$$\leqq p|z|^{p-1} + |z|^p\sum_{n=1}^{\infty}(p+n)|a_{p+n}|$$
$$\leqq p|z|^{p-1} + \frac{p(p+1)(1-\alpha)}{p(1-\alpha)+1}|z|^p$$

and

$$(2.15)\qquad |f'(z)| \geqq p|z|^{p-1} - \sum_{n=1}^{\infty}(p+n)|a_{p+n}||z|^{p+n-1}$$
$$\geqq p|z|^{p-1} - |z|^{p}\sum_{n=1}^{\infty}(p+n)|a_{p+n}|$$
$$\geqq p|z|^{p-1} - \frac{p(p+1)(1-\alpha)}{p(1-\alpha)+1}|z|^{p}$$

for $z \in \mathcal{U}$ with the aid of (2.13).

COROLLARY 2. *Under the hypotheses of* Theorem 3, $f'(z)$ *is included in a disk with its center at the origin and radius* r *given by*

$$(2.16)\qquad r = p + \frac{p(p+1)(1-\alpha)}{p(1-\alpha)+1}$$

3. Application of the fractional calculus,

We give the distortion theorems for the fractional calculus of functions $f(z)$ in $\mathcal{P}_p^*(\alpha)$.

THEOREM 4. *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $\mathcal{P}_p^*(\alpha)$. *Then we have*

$$(3.1)\qquad \frac{\Gamma(p+1)}{\Gamma(p+1+\lambda)}|z|^{p+\lambda}\left\{1-\frac{p(1-\alpha)}{p(1-\alpha)+1}|z|\right\} \leqq |D_z^{-\lambda}f(z)|$$
$$\leqq \frac{\Gamma(p+1)}{\Gamma(p+1+\lambda)}|z|^{p+\lambda}\left\{1+\frac{p(1-\alpha)}{p(1-\alpha)+1}|z|\right\}$$

for $\lambda > 0$ *and* $z \in \mathcal{U}$. *Further*

$$(3.2)\qquad \frac{\Gamma(p+1)}{\Gamma(p+1-\lambda)}|z|^{p-\lambda}\left\{1-\frac{(p+1)(1-\alpha)}{p(1-\alpha)+1}|z|\right\} \leqq |D_z^{\lambda}f(z)|$$
$$\leqq \frac{\Gamma(p+1)}{\Gamma(p+1-\lambda)}|z|^{p-\lambda}\left\{1+\frac{(p+1)(1-\alpha)}{p(1-\alpha)+1}|z|\right\}$$

for $0 < \lambda < 1$ *and* $z \in \mathcal{U}$.

Proof. Since $f(z)$ is in the class $\mathcal{P}_p^*(\alpha)$, $f(z)$ satisfies (2.6). Now, we consider the function

$$(3.3)\qquad F(z) = \frac{\Gamma(p+1+\lambda)}{\Gamma(p+1)}z^{-\lambda}D_z^{-\lambda}f(z)$$
$$= z^{p} + \sum_{n=1}^{\infty}\frac{\Gamma(p+n+1)\Gamma(p+1+\lambda)}{\Gamma(p+n+1+\lambda)\Gamma(p+1)}a_{p+n}z^{p+n}$$

for $\lambda>0$. Note that

$$0<\frac{\Gamma(p+n+1)\Gamma(p+1+\lambda)}{\Gamma(p+n+1+\lambda)\Gamma(p+1)}<1 \tag{3.4}$$

for $\lambda>0$ and $n\geqq 1$. Therefore, by using (2.6) and (3.4), we obtain

$$\begin{aligned}|F(z)|&=\left|\frac{\Gamma(p+1+\lambda)}{\Gamma(p+1)}z^{-\lambda}D_z^{-\lambda}f(z)\right| \\ &\leqq |z|^p+|z|^{p+1}\sum_{n=1}^{\infty}\frac{\Gamma(p+n+1)\Gamma(p+1+\lambda)}{\Gamma(p+n+1+\lambda)\Gamma(p+1)}|a_{p+n}| \\ &\leqq |z|^p+|z|^{p+1}\sum_{n=1}^{\infty}|a_{p+n}| \\ &\leqq |z|^p+\frac{p(1-\alpha)}{p(1-\alpha)+1}|z|^{p+1}\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}|F(z)|&=\left|\frac{\Gamma(p+1+\lambda)}{\Gamma(p+1)}z^{-\lambda}D_z^{-\lambda}f(z)\right| \\ &\geqq |z|^p-|z|^{p+1}\sum_{n=1}^{\infty}\frac{\Gamma(p+n+1)\Gamma(p+1+\lambda)}{\Gamma(p+n+1+\lambda)\Gamma(p+1)}|a_{p+n}| \\ &\geqq |z|^p-|z|^{p+1}\sum_{n=1}^{\infty}|a_{p+n}| \\ &\geqq |z|^p-\frac{p(1-\alpha)}{p(1-\alpha)+1}|z|^{p+1}\end{aligned} \tag{3.6}$$

for $\lambda>0$ and $z\in\mathcal{U}$. Thus (3.1) follows from (3.5) and (3.6).

Next, we consider the following function

$$\begin{aligned}G(z)&=\frac{\Gamma(p+1-\lambda)}{\Gamma(p+1)}z^{\lambda}D_z^{\lambda}f(z) \\ &=z^p+\sum_{n=1}^{\infty}\frac{\Gamma(p+n+1)\Gamma(p+1-\lambda)}{\Gamma(p+n+1-\lambda)\Gamma(p+1)}a_{p+n}z^{p+n}.\end{aligned} \tag{3.7}$$

Then it is easy that

$$\sum_{n=1}^{\infty}\left(\frac{p+n}{p}\right)|a_{p+n}|\leqq\frac{(p+1)(1-\alpha)}{p(1-\alpha)+1} \tag{3.8}$$

by (2.13) and

$$1<\frac{\Gamma(p+n+1)\Gamma(p+1-\lambda)}{\Gamma(p+n+1-\lambda)\Gamma(p+1)}<\frac{p+n}{p} \tag{3.9}$$

for $0<\lambda<1$ and $n\geqq 1$. Consequently we can show that

$$(3.10)\quad |G(z)|=\left|\frac{\Gamma(p+1-\lambda)}{\Gamma(p+1)}z^{\lambda}D_z^{\lambda}f(z)\right|$$
$$\leqq |z|^p+|z|^{p+1}\sum_{n=1}^{\infty}\frac{\Gamma(p+n+1)\Gamma(p+1-\lambda)}{\Gamma(p+n+1-\lambda)\Gamma(p+1)}|a_{p+n}|$$
$$\leqq |z|^p+|z|^{p+1}\sum_{n=1}^{\infty}\left(\frac{p+n}{p}\right)|a_{p+n}|$$
$$\leqq |z|^p+\frac{(p+1)(1-\alpha)}{p(1-\alpha)+1}|z|^{p+1}$$

and

$$(3.11)\quad |G(z)|=\left|\frac{\Gamma(p+1-\lambda)}{\Gamma(p+1)}z^{\lambda}D_z^{\lambda}f(z)\right|$$
$$\geqq |z|^p-|z|^{p+1}\sum_{n=1}^{\infty}\frac{\Gamma(p+n+1)\Gamma(p+1-\lambda)}{\Gamma(p+n+1-\lambda)\Gamma(p+1)}|a_{p+n}|$$
$$\geqq |z|^p-|z|^{p+1}\sum_{n=1}^{\infty}\left(\frac{p+n}{p}\right)|a_{p+n}|$$
$$\geqq |z|^p-\frac{(p+1)(1-\alpha)}{p(1-\alpha)+1}|z|^{p+1}$$

for $0<\lambda<1$ and $z\in\mathcal{U}$. The second half of the theorem is given by (3.10) and (3.11). Thus we have the theorem.

COROLLARY 3. *Under the hypotheses of* Theorem 4, $D_z^{-\lambda}f(z)$ *is included in a disk with its center at the origin and radius r given by*

$$(3.12)\qquad r=\frac{\Gamma(p+1)}{\Gamma(p+1+\lambda)}\left\{1+\frac{p(1-\alpha)}{p(1-\alpha)+1}\right\},$$

and $D_z^{\lambda}f(z)$ *is included in a disk with its center at the origin and radius R given by*

$$(3.13)\qquad R=\frac{\Gamma(p+1)}{\Gamma(p+1-\lambda)}\left\{1+\frac{(p+1)(1-\alpha)}{p(1-\alpha)+1}\right\}.$$

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