J. Korean Math. Soc. **21** (1984), *No.* 1, 49–61

ON *CR*-SUBMANIFOLDS OF LOCALLY CONFORMAL KÄHLER MANIFOLDS

KOJI MATSUMOTO

0. Introduction

Recently, A. Bejancu [1, 2] introduced the notion of a *CR*-submanifold of a Kähler manifold and B. Y. Chen [4, 5], D. E. Blair and B. Y. Chen [3] and K. Yano and M. Kon [14, 15] had a lot of very interesting results of this submanifold.

On the other hand, T. Kashiwada [7, 8, 9], S. Tachibana [11] and I. Vaisman [12, 13] studied locally conformal Kähler manifolds and the author [10] considered submanifolds of locally conformal Kähler manifolds.

In this paper, we shall consider *CR*-submanifolds of locally conformal Kähler manifolds.

1. Preliminaries

Let $\tilde{M}(J, \langle\, ,\rangle, \alpha)$ be a locally conformal Kähler manifold (an l. c. K-manifold). Then, by definition, at any point of $\tilde{M}$, there exists a neighborhood in which a conformal metric $\langle\, ,\rangle^* = e^{-2\rho}\langle\, ,\rangle$ is a Kähler one, i. e.,

$$\nabla^*(e^{-2\rho}J)=0, \quad d\rho=\alpha,$$

where ∇^* is the covariant differentiation with respect to $\langle\, ,\rangle^*$. By virtue of the above equation, we get

$$(1.1)\quad (\tilde{\nabla}_Z\phi)(Y,X) = -\langle\alpha, Y\rangle\phi(Z,X) + \langle X, J\alpha\rangle\langle Z, Y\rangle - \langle\alpha, X\rangle\phi(Z,Y) - \phi(\alpha, Y)\langle Z, X\rangle,$$

where $\tilde{\nabla}$ is the covariant differentiation with respect to $\langle\, ,\rangle$,

$$(1.2)\quad \phi(X,Y) = \langle JX, Y\rangle$$

and we write $\alpha(X)=\langle\alpha, X\rangle$ for any vector fields X, Y and Z on $\tilde{M}$.

Received October 20, 1983.

The following proposition is well-known [7];

PROPOSITION 1. 1. *A Hermitian manifold $\tilde{M}(J, \langle , \rangle)$ is an l. c. K-manifold if and only if there exists a global 1-form α satisfying*

$$(1.3)\quad (\tilde{\nabla}_Z\phi)(Y, X) = -\langle\beta, Y\rangle\langle Z, X\rangle + \langle\beta, X\rangle\langle Z, Y\rangle - \langle\alpha, Y\rangle\phi(Z, X) + \langle\alpha, X\rangle\phi(Z, Y),$$

$$(1.4)\quad (\tilde{\nabla}_Y\alpha)X = (\tilde{\nabla}_X\alpha)Y,$$

where $\langle\beta, X\rangle = \langle J\alpha, X\rangle$.

In an l. c. K-manifold $\tilde{M}$, we define a symmetric tensor field $P(X, Y)$ as

$$(1.5)\quad P(Y, X) = -(\tilde{\nabla}_Y\alpha)X - \alpha(Y)\alpha(X) + \frac{1}{2}\|\alpha\|^2\langle Y, X\rangle,$$

where $\|\alpha\|$ denotes the length of the Lee form α with respect to $\langle , \rangle$.

In this paper, we assume that the tensor field P is hybrid, that is,

$$(1.6)\quad P(Y, JX) + P(JY, X) = 0.$$

An l. c. K-manifold $\tilde{M}$ is called an *l. c. K-space form* if it has a constant holomorphic sectional curvature H. Then the Riemannian curvature tensor $\tilde{R}$ of an l. c. K-space form $\tilde{M}(H)$ with constant holomorphic sectional curvature H is given by [7]

$$\begin{aligned}(1.7)\quad 4\tilde{R}(X, Y; Z, W) = {} & H\{\langle X, W\rangle\langle Y, Z\rangle - \langle X, Z\rangle\langle Y, W\rangle + \\ & +\phi(X, W)\phi(Y, Z) - \phi(X, Z)\phi(Y, W) - 2\phi(X, Y)\phi(Z, W)\} \\ & +3\{P(X, W)\langle Y, Z\rangle - P(X, Z)\langle Y, W\rangle + \langle X, W\rangle P(Y, Z) \\ & -\langle X, Z\rangle P(Y, W)\} - \tilde{P}(X, W)\phi(Y, Z) + \tilde{P}(X, Z)\phi(Y, W) \\ & -\phi(X, W)\tilde{P}(Y, Z) + \phi(X, Z)\tilde{P}(Y, W) + 2\{\tilde{P}(X, Y)\phi(Z, W) \\ & +\phi(X, Y)\tilde{P}(Z, W)\},\end{aligned}$$

where

$$(1.7)\quad \tilde{P}(X, Y) = -P(X, JY).$$

2. Submanifolds of an l. c. K-manifold

Let $\tilde{M}(J, \langle , \rangle, \alpha)$ be a complex m-dimensional l. c. K-manifold and M be a real n-dimensional Riemannian manifold isometrically immersed in $\tilde{M}$. We denote by the same $\langle , \rangle$ the metric tensor induced on M. Let ∇ be the covariant differentiation with respect to the induced metric on M. Then the Gauss and Weingarten formulas for M are respectively given by

$$\tilde{\nabla}_Y X = \nabla_Y X + \sigma(Y, X), \tag{2.1}$$

$$\tilde{\nabla}_X \xi = -A_\xi X + \nabla^\perp_X \xi \tag{2.2}$$

for any vector fields Y and X tangent to M and any vector field ξ normal to M, where σ denotes the second fundamental form and $\nabla^\perp$ the linear connection, called the normal connection, induced in the normal bundle $T^\perp M$. The second fundamental tensor A_ξ is related to σ by

$$\langle A_\xi X, Y\rangle = \langle \sigma(X, Y), \xi\rangle. \tag{2.3}$$

For any vector field U tangent to M, we put

$$JU = TU + QU, \tag{2.4}$$

where TU and QU are the tangential and the normal components of JU, respectively. Then T is an endomorphism of the tangent bundle TM and Q is a normal-bundle-valued 1-form on TM.

For any vector field ξ normal to M, we put

$$J\xi = t\xi + f\xi, \tag{2.5}$$

where $t\xi$ and $f\xi$ are the tangential and the normal components of $J\xi$, espectively. Then f is an endomorphism of $T^\perp M$ and t is a tangent-bundle-valued 1-form on $T^\perp M$.

For the second fundamental form σ, we define the covariant differentiation with respect to the connection on $TM \oplus T^\perp M$ by

$$(\bar{\nabla}_X \sigma)(Y, Z) = \nabla^\perp_X(\sigma(Y, Z)) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z) \tag{2.6}$$

for any vector fields X, Y and Z tangent to M[6]. Then the equations of Gauss and Codazzi are respectively given by

$$R(X, Y; Z, W) = \tilde{R}(X, Y; Z, W)) + \langle \sigma(X, W), \sigma(Y, Z)\rangle - \langle \sigma(X, Z), \sigma(Y, W)\rangle, \tag{2.7}$$

$$(\tilde{R}(X, Y)Z)^\perp = (\bar{\nabla}_X \sigma)(Y, Z) - (\bar{\nabla}_Y \sigma)(X, Z) \tag{2.8}$$

for any vector fields X, Y, Z and W tangent to M, where R denotes the Riemannian curvature tensor with respect to the induced metric on M and $(\tilde{R}(X, Y)Z)^\perp$ means the normal component of $\tilde{R}(X, Y)Z$.

3. *CR*-submanifolds of an l. c. K-manifold.

In this section, we shall define a *CR*-submanifold of an l. c. K-manifold and give several fundamental properties of this submanifold.

DEFINITION 3. 1. A submanifold M of an l. c. K-manifold $\tilde{M}$ is called a *CR-submanifold* if there exists a differentiable distribution D : $x \to D_x \subset T_xM$ on M satisfying the following conditions;

(i) D is holomorphic, i. e., $JD_x = D_x$ for each $x \in M$ and

(ii) the complementary orthogonal distribution $D^\perp : x \to D_x^\perp \subset T_xM$ is totally real, i. e., $JD_x^\perp \subset T_x^\perp M$ for each $x \in M$.

If $\dim D_x^\perp = 0$ (resp. $\dim D_x = 0$), then the *CR*-submanifold is called a *holomorphic* (resp. a *totally real*) *submanifold*. If $\dim D_x = \dim T_xM$ then the *CR*-submanifold is called an *anti-holomorphic submanifold* or a *generic submanifold* [15]. A *CR*-submanifold is called a *proper CR-submanifold* if it is neither holomorphic nor totally real.

It is easy show that T defined in (2. 4) is an almost complex structure of D, i. e., $T^2X = -X$ for any X in D. Hence the dimension of D_x for each $x \in M$ is even.

We denote by ν the complementary orthogonal subbundle of $JD^\perp$ in $T^\perp M$. Then we have

$$(3.1) \quad T^\perp M = JD^\perp \oplus \nu, \quad JD^\perp \perp \nu.$$

For any vector field U tangent to M and Z in $D^\perp$, we have

$$(3.2) \quad \tilde{\nabla}_U JZ = -A_{JZ}U + \nabla^\perp_U JZ.$$

By virtue of (1. 3) and (3. 2), we obtain

$$(3.3) \quad \nabla_U Z + \sigma(U, Z) = -\langle \beta, Z\rangle JU - \langle U, Z\rangle \alpha + \langle \alpha, Z\rangle U + JA_{JZ}U - J\nabla^\perp_U JZ.$$

From this, we get

$$(3.4) \quad \langle \nabla_U Z, X\rangle = -\langle \beta, Z\rangle\langle JU, X\rangle - \langle U, Z\rangle\langle \alpha, X\rangle + \langle \alpha, Z\rangle\langle U, X\rangle + \langle JA_{JZ}U, X\rangle$$

for any vector field U tangent to M, X in D and Z in $D^\perp$. In (3. 4), if U is an element of $D^\perp$ (put it W), then (3. 4) can be written as

(3.5) $\langle \nabla_W Z, X\rangle = -\langle \alpha, X\rangle\langle Z, W\rangle + \langle JA_{JZ}W, X\rangle$.

Thus we have

(3.6) $\langle [W, Z], X\rangle = \langle J(A_{JZ}W - A_{JW}Z), X\rangle$

where $[Z, W] = \nabla_Z W - \nabla_W Z$.

Next, we can easily show the following;

(3.7) $A_{JZ}W - A_{JW}Z = \langle \beta, Z\rangle W - \langle \beta, W\rangle Z$

for any Z and W in $D^\perp$. Thus we have from (3.6) and (3.7)

PROPOSITION 3.1. *In a CR-submanifold M of an l.c. K-manifold $\tilde{M}$, the distribution $D^\perp$ is integrable.*

For any vector field U tangent to M, X in D and ξ in ν, we can prove

$$\langle A_\xi JX, U\rangle = \langle \beta, \xi\rangle\langle X, U\rangle - \langle \alpha, \xi\rangle\langle JX, U\rangle - \langle A_{J\xi}X, U\rangle,$$

that is,

(3.8) $$A_\xi JX + A_{J\xi}X = \langle \beta, \xi\rangle X - \langle \alpha, \xi\rangle JX.$$

Now, we assume that the distribution D is integrable. Then for any X and Y in D, $[X, Y]$ is an element of D, that is, $\langle [X, Y], Z\rangle = 0$ for any Z in $D^\perp$. From this equation we obtain

(3.9) $$\langle \sigma(X, JY) - \sigma(Y, JX) - 2\langle JX, Y\rangle\alpha, JZ\rangle = 0.$$

Conversely if (3.9) is satisfied, then it is easily seen that the distribution D is integrable. Thus we have

PROPOSITION 3.2. *The distribution D of a CR-submanifold of an l.c. K-manifold is integrable if and only if* (3.9) *is satisfied.*

Let the leaf $M^\perp$ of the distribution $D^\perp$ be totally geodesic in M, that is, $\nabla_Z W$ is an element of $D^\perp$ for any Z and W in $D^\perp$. This means $\langle \nabla_Z W, X\rangle = 0$ for any X in D. From this, we can prove

PROPOSITION 3.3. *The leaf $M^\perp$ of the distribution $D^\perp$ of a CR-submanifold M of an l.c. K-manifold $\tilde{M}$ is totally geodesic in M if and only if*

(3.10) $$\langle A_{JW}Z + \langle Z, W\rangle\beta, X\rangle = 0$$

for any X in D and Z and W in $D^\perp$.

Next, we shall prove

PROPOSITION 3.4. *If, in a CR-submanifold M of an l. c. K-manifold $\tilde{M}$, the distribution D is integrable and the leaf $M^{\perp}$ of $D^{\perp}$ is totally geodesic in M, then we have*

$$A_{JZ}JX+JA_{JZ}X-2\langle\alpha, JZ\rangle JX+\langle\alpha, X\rangle Z+\langle\beta, X\rangle JZ=0 \tag{3.11}$$

for any X in D and Z in $D^{\perp}$.

Proof. To prove (3.11), it is sufficient that we show

$$\langle A_{JZ}JX+JA_{JZ}X-2\langle\alpha, JZ\rangle JX+\langle\alpha, X\rangle Z+\langle\beta, X\rangle JZ, D\oplus D^{\perp}\oplus JD^{\perp}\rangle=\{0\}.$$

By virtue of (3.9) and (3.10), we can show the above equation.

4. Covariant differentiations

Let T, f, Q and t be the endomorphisms and the vector-bundle-valued 1-forms defined in (2.4) and (2.5), respectively. Let us define the covariant differentiation of T, Q, t and f as follows;

$$(\bar{\nabla}_U T)V=\nabla_U(TV)-T\nabla_U V, \tag{4.1}$$

$$(\bar{\nabla}_U Q)V=\nabla^{\perp}_U(QV)-Q\nabla_U V, \tag{4.2}$$

$$(\bar{\nabla}_U t)\xi=\nabla_U(t\xi)-t\nabla^{\perp}_U\xi, \tag{4.3}$$

$$(\bar{\nabla}_U f)\xi=\nabla^{\perp}_U(f\xi)-f\nabla^{\perp}_U\xi \tag{4.4}$$

for any vector fields U and V tangent to M and any vector field ξ normal to M.

The endomorphism T (resp. the endomorphism f, the vector-bundle-valued 1-forms Q or t) is *parallel* if $\bar{\nabla}T=0$ (resp. $\bar{\nabla}f=0$, $\bar{\nabla}Q=0$ or $\bar{\nabla}t=0$).

From (1.1), (2.2) and (2.5), we can easily show

$$(\bar{\nabla}_U T)V=A_{QV}U+t\sigma((U, V)-\langle\beta, V\rangle U+\langle U, V\rangle\beta_1 -\langle\alpha, V\rangle TU+\langle TU, V\rangle\alpha_1, \tag{4.5}$$

$$(\bar{\nabla}_U Q)V=-\sigma(U, TV)+\langle U, V\rangle\beta_2-\langle\alpha, V\rangle QU+\langle TU, V\rangle\alpha_2 +f\sigma(U, V), \tag{4.6}$$

$$(\bar{\nabla}_U t)\xi=A_{f\xi}U-\langle\beta, \xi\rangle U-\langle\alpha, \xi\rangle TU+\langle QU, \xi\rangle\alpha_1-TA_{\xi}U, \tag{4.7}$$

$$(\bar{\nabla}_U f)\xi=-\sigma(U, t\xi)-\langle\alpha, \xi\rangle QU+\langle QU, \xi\rangle\alpha_2-QA_{\xi}U \tag{4.8}$$

for any vector fields U and V tangent to M and any vector field ξ normal to M, where α_1, β_1 and α_2, β_2 are the tangential and the normal components of α and β, respectively.

5. CR-products of an l. c. K-manifold

DEFINITION 5. 1. A CR-submanifold M of an l. c. K-manifold $\tilde{M}$ is called a *CR-product* if it is locally a Riemannian product of a holomorphic submanifold M^I and a totally real submanifold $M^{\perp}$ of M.

In this section, our main object is to prove the following;

THEOREM 5. 1. *A CR-submanifold M of an l. c. K-manifold $\tilde{M}$ is a CR-product if and only if the endomorphism T is parallel.*

Proof. Let M be a CR-submanifold of an l. c. K-manifold $\tilde{M}$ and T be parallel. Then we have from (4. 5)

$$A_{QV}U+t\sigma(U, V)-\langle\beta, V\rangle U+\langle U, V\rangle\beta_1-\langle\alpha, V\rangle TU +\langle TU, V\rangle\alpha_1=0 \tag{5.1}$$

for any vector fields U and V tangent to M. In (5. 1), if the vector field V is in D (put it X), then (5. 1) is written as

$$t\sigma(U,X)-\langle\beta,X\rangle U+\langle U,X\rangle\beta_1-\langle\alpha,X\rangle TU+\langle TU,X\rangle\alpha_1=0. \tag{5.2}$$

From this, we get

$$\langle t\sigma(U,X),Z\rangle-\langle\beta,X\rangle\langle U,Z\rangle+\langle\beta,Z\rangle\langle U,X\rangle-\langle\alpha,X\rangle\langle TU,Z\rangle +\langle TU,X\rangle\langle\alpha,Z\rangle=0$$

for any vector field U tangent to M, X in D and Z in $D^{\perp}$. This equation means

$$A_{JZ}X=-\langle\beta,X\rangle Z+\langle\beta,Z\rangle X-\langle\alpha,Z\rangle JX \tag{5.3}$$

for any X in D and Z in $D^{\perp}$. By virtue of (5. 3), we obtain

$$\langle A_{JZ}X, Y\rangle=\langle\beta,Z\rangle\langle Y,X\rangle-\langle\alpha,Z\rangle\langle JX, Y\rangle \tag{5.4}$$

for any X and Y in D and Z in $D^{\perp}$. Since the second fundamental form σ is symmetric, we have from (5. 4) $\langle\alpha,Z\rangle=0$ if $D\neq\{0\}$. Taking account of (3. 3), we have

$$JA_{JZ}X=J\nabla_X JZ+\langle\beta,Z\rangle JX+\nabla_X Z+\sigma(X,Z)$$

for any X in D and Z in $D^{\perp}$. Thus, by virtue of (5. 4) and the above equation, we have $\langle\nabla_X JY,Z\rangle=0$ for any X and Y in D and Z in $D^{\perp}$. This means the leaf $M^{\top}$ of D is totally geodesic in M. Of course, we can easily see the distribution D is integrable.

Next, we have from (5. 1)

$$A_{QW}Z+t\sigma(Z, W)-\langle\beta, W\rangle Z+\langle Z, W\rangle\beta_1=0$$

for any Z and W in $D^{\perp}$. From this, we get $\langle A_{JW}Z+\langle Z, W\rangle\beta, X\rangle=0$ for any X in D. Thus by virtue of Proposition 3. 3, the leaf $M^{\perp}$ of $D^{\perp}$ is totally geodesic in M.

Conversely, in a CR-product of an l. c. K-manifold, it is trivial that the endomorphism T is parallel.

From (5. 3), we have

PROPOSITION 5. 2. *A CR-submanifold M of an* l. c. K-*manifold $\tilde{M}$ is a CR-product if and only if*

$$A_{JZ}X=-\langle\beta, X\rangle Z+\langle\beta, Z\rangle X-\langle\alpha, Z\rangle JX \tag{5.5}$$

for any X in D and Z in $D^{\perp}$.

Let M be a CR-product of an l. c. K-manifold $\tilde{M}$. Let us calculate the holomorphic bisectional curvature $\tilde{H}_B(X, Z)$ for any units X in D and Z in $D^{\perp}$, where $\tilde{H}_B(X, Z)$ is defined by

$$\tilde{H}_B(X, Z)=\tilde{R}(X, JX; JZ, Z). \tag{5.6}$$

By the straightfoward calculation, we get

$$\begin{aligned}\tilde{H}_B(X, Z)=&2\|\sigma(X, Z)\|^2+2\|\langle\beta, X\rangle\|^2+2\|\langle\alpha, X\rangle\|^2\\&-2\|\alpha\|^2+\langle\tilde{\nabla}_X\alpha, X\rangle+\langle\tilde{\nabla}_{JX}\alpha, JX\rangle.\end{aligned} \tag{5.7}$$

Let the ambient manifold $\tilde{M}$ be an l. c. K-space form $\tilde{M}(H)$ with constant holomorphic sectional curvature H. Then we obtain

$$\tilde{H}_B(X, Z)=\frac{1}{2}(H+\langle\tilde{\nabla}_X\alpha, X\rangle+\langle\tilde{\nabla}_Z\alpha, Z\rangle-\|\langle\alpha, X\rangle\|^2-\|\alpha\|^2). \tag{5.8}$$

Substituting (5. 8) into (5. 7), we have

$$\begin{aligned}H+3\|\alpha\|^2=&4\|\sigma(X, Z)\|^2+4\|\langle\beta, X\rangle\|^2+5\|\langle\alpha, X\rangle\|^2\\&+\langle\tilde{\nabla}_X\alpha, X\rangle-\langle\tilde{\nabla}_Z\alpha, Z\rangle+2\langle\tilde{\nabla}_{JX}\alpha, JX\rangle.\end{aligned} \tag{5.9}$$

Thus we have

PROPOSITION 5. 3. *In a CR-product of an l. c. K-space form $\tilde{M}(H)$, we have* (5. 9).

6. CR-submanifolds with $\bar{\nabla}Q=0$

In this section, at first, we shall prove

PROPOSITION 6. 1. *Let M be a CR-submanifold of an l. c. K-manifold $\tilde{M}$ Then $\bar{\nabla}Q=0$ if and only if $\bar{\nabla}t=0$.*

Proof. Let t be parallel. Then, by virtue of (4. 7), we have

$$\langle A_{f\xi}U, V\rangle=\langle\beta,\xi\rangle\langle U, V\rangle+\langle\alpha,\xi\rangle\langle TU, V\rangle-\langle QU,\xi\rangle\langle\alpha, V\rangle \\ +\langle TA_{\xi}U, V\rangle$$

for any vector fields U and V tangent to M and any vector field ξ normal to M. The above equation means

$$\langle\sigma(U, V), f\xi\rangle=\langle U, V\rangle\langle\beta,\xi\rangle+\langle TU, V\rangle\langle\alpha,\xi\rangle-\langle\alpha, V\rangle\langle QU,\xi\rangle \\ -\langle\sigma(U, TV),\xi\rangle,$$

which is equivalent to

$$f\sigma(U, V)-\sigma(U, TV)+\langle U, V\rangle\beta_2+\langle TU, V\rangle\alpha_2 \\ -\langle\alpha, V\rangle QU=0, \tag{6. 1}$$

i. e., $\bar{\nabla}Q=0$.

Next, we shall prove

THEOREM 6. 2. *Let M be a CR-submanifold of an l. c. K-manifold $\tilde{M}$. If Q is parallel, then we have*

(i) the submanifold M is a CR-product,

(ii) $A_{\nu}D^{\perp}\subset D^{\perp}$, and

(iii) $\sigma(X, Z)=-\langle\beta, X\rangle JZ$ for any X in D and Z in $D^{\perp}$.

Proof. Let Q be parallel. Then we have (6. 1). Thus, for any Z in $D^{\perp}$, we get

$$-\langle\sigma(V, TX), JZ\rangle+\langle X, V\rangle\langle\beta_2, JZ\rangle+\langle TV, X\rangle\langle\alpha_2, JZ\rangle \\ -\langle\alpha, X\rangle\langle QV, JZ\rangle=0,$$

from which,

$$A_{JZ}X=-\langle\alpha, Z\rangle JX+\langle\beta, Z\rangle X-\langle\beta, X\rangle Z$$

for any X in D and Z in $D^{\perp}$. By virtue of Proposition 5. 2, the above equation tells us that the submanifold M is a CR-product.

In (6. 1), if the vector field U is in $D^{\perp}$ (put it W), then we have

$$f\sigma(V, W)+\langle V, W\rangle\beta_2+\langle TV, W\rangle\alpha_2=0.$$

The above equation means

$$\langle f\sigma(V, W), J\xi\rangle+\langle V, W\rangle\langle\beta_2, J\xi\rangle=0$$

for any ξ in ν. From this, we get

$$A_\xi W=-\langle\alpha, \xi\rangle W \tag{6.2}$$

for any W in $D^\perp$ and ξ in ν. This equation means (ii).

In (6.1), if we put $U=X\in D$ and $V=Z\in D^\perp$, then we obtain $f\sigma(X, Z)=0$. Moreover, in (6.1), if we put $U=Z\in D^\perp$ and $V=X\in D$, then we have

$$-\sigma(Z, JX)-\langle\alpha, X\rangle JZ+f\sigma(X, Z)=0.$$

Thus we have from the above two equations (iii).

Now, let M be a CR-submanifold satisfying $\bar{\nabla}Q=0$ of an l. c. K-space form $\tilde{M}(H)$. Then, by virtue of (iii) in the above theorem, the equation (5.9) can be written as

$$\begin{aligned} H+3\|\alpha\|^2=8\|\langle\alpha, X\rangle\|^2+5\|\langle\beta, X\rangle\|^2+\langle\tilde{\nabla}_X\alpha, X\rangle \\ -\langle\tilde{\nabla}_Z\alpha, Z\rangle+2\langle\tilde{\nabla}_{JX}\alpha, JX\rangle \end{aligned} \tag{6.3}$$

for any units X in D and Z in $D^\perp$. Thus, in our submanifold M, if $\dim D_x>2$ for each x in M, then we can take a X in D as $\langle\alpha, X\rangle=0$ and $\langle\beta, X\rangle=0$. Then the equation (6.3) can be written as

$$H+3\|\alpha\|^2=-\langle\alpha, \tilde{\nabla}_X X\rangle+\langle\alpha, \tilde{\nabla}_Z Z\rangle-2\langle\alpha, \tilde{\nabla}_{JX}JX\rangle. \tag{6.4}$$

In the above equation, let us replace X by JX. Then we have

$$H+3\|\alpha\|^2=-\langle\alpha, \tilde{\nabla}_{JX}JX\rangle+\langle\alpha, \tilde{\nabla}_Z Z\rangle-2\langle\alpha, \tilde{\nabla}_X X\rangle.$$

By virtue of the above equation and (6.4), we have

$$\langle\alpha, \tilde{\nabla}_X X\rangle=\langle\alpha, \tilde{\nabla}_{JX}JX\rangle. \tag{6.5}$$

Thus we have

$$H+3\|\alpha\|^2=-3\langle\alpha, \tilde{\nabla}_X X\rangle+\langle\alpha, \tilde{\nabla}_Z Z\rangle \tag{6.6}$$

for any units X in D such that $\langle\alpha, X\rangle=0$ and Z in $D^\perp$. Thus we have

THEOREM 6.3. *In a CR-submanifold M with $\bar{\nabla}Q=0$ of an l. c. K-space form $\tilde{M}(H)$, if there are two vector fields X in D and Z in $D^\perp$ satisfying the conditions*; (*i*) $\langle\alpha, X\rangle=0$, $\tilde{\nabla}_X X=0$, (ii) $\tilde{\nabla}_Z Z=0$, *then* *the length $\|\alpha\|$ of the Lee form α is constant and the holomorphic sectional curvature H is non-positive.*

7. Mixed foliate *CR*-submanifolds

DEFINITION 7.1. A CR-submanifold M of an l. c. K-manifold $\tilde{M}$ is said to be *mixed foliate* if

(i) the distribution D is integrable and

(ii) $\sigma(D, D^{\perp})=\{0\}$.

For a mixed foliate CR-submanifold of an l. c. K-manifold we shall prove the following;

PROPOSITION 7.1. *Let M be a mixed foliate CR-submanifold of an l. c. K-manifold $\tilde{M}$. Then for any units X in D and Z in $D^{\perp}$ we have*

$$\tilde{H}_B(X, Z)-2\langle\beta, Z\rangle^2=-\|A_{JZ}X\|^2-\|A_{JZ}JX\|^2. \tag{7.1}$$

Proof. If M is a mixed foliate CR-submanifold, then we have

$$\sigma(D, D^{\perp})=\{0\},\ [D, D]\subset D \text{ and } \langle\sigma(X, JY)-\sigma(Y, JX)-2\langle JX, Y\rangle\alpha, JZ\rangle=0 \tag{7.2}$$

for any X and Y in D and any Z in $D^{\perp}$. By the equation of Codazzi and (2.6), we have

$$\tilde{H}_B(X, Z)=-\tilde{R}(X, JX; Z, JZ)=\langle\sigma(\nabla_X JX, Z)+\sigma(JX, \nabla_X Z)-\sigma(\nabla_{JX}X, Z)-\sigma(X, \nabla_{JX}Z), JZ\rangle$$

for any units X in D and Z in $D^{\perp}$.

On the other hand, since

$$\langle\sigma(\nabla_X JX, Z)-\sigma(\nabla_{JX}X, Z), JZ\rangle=\langle A_{JZ}\nabla_X JX-A_{JZ}\nabla_{JX}X, Z\rangle$$
$$=\langle A_{JZ}[X, JX], Z\rangle=\langle\sigma([X, JX], Z, JZ\rangle=0 \text{ by } (7.2)_{1,2},$$

we have

$$\begin{aligned}\tilde{H}_B(X, Z)&=\langle\sigma(JX, \nabla_X Z)-\sigma(X, \nabla_{JX}Z), JZ\rangle\\&=\langle A_{JZ}JX, \nabla_X Z\rangle-\langle A_{JZ}X, \nabla_{JX}Z\rangle.\end{aligned} \tag{7.3}$$

Substituting (3.3) into (7.3), we get

$$\begin{aligned}\tilde{H}_B(X, Z)=&-\langle\beta, Z\rangle\{\langle A_{JZ}JX, JX\rangle+\langle A_{JZ}X, X\rangle\}\\&+\langle\alpha, Z\rangle\{\langle A_{JZ}JX, X\rangle-\langle A_{JZ}X, JX\rangle\}-2\langle A_{JZ}X, JA_{JZ}JX\rangle.\end{aligned} \tag{7.4}$$

Furthermore, by virtue of (7.2), we obtain

(7.5) $\tilde{H}_B(X,Z) = -2\langle\beta, Z\rangle^2 - 2\|A_{JZ}JX\|^2 - 4\langle JX, A_{JZ}JX\rangle\langle\alpha, JZ\rangle.$

Since $\tilde{H}_B(X,Z) = \tilde{H}_B(JX,Z)$, we have from (7.5)

(7.6) $\tilde{H}_B(X,Z) = -2\langle\beta, Z\rangle^2 - 2\|A_{JZ}X\|^2 + 4\langle X, A_{JZ}X\rangle\langle\beta, Z\rangle.$

By virtue of $\langle A_{JZ}JX, JX\rangle = -\langle A_{JZ}X, X\rangle - 2\langle\alpha, JZ\rangle$, (7.5) can be written as

$$\tilde{H}_B(X,Z) = 6\langle\beta, Z\rangle^2 - 2\|A_{JZ}JX\|^2 + 4\langle A_{JZ}X, X\rangle\langle\alpha, JZ\rangle.$$

We have from (7.6) and the above equation

(7.7) $4\langle A_{JZ}X, X\rangle\langle\beta, Z\rangle = -\tilde{H}_B(X,Z) + 6\langle\beta, Z\rangle^2 - 2\|A_{JZ}JX\|^2.$

Substituting (7.7) into (7.6), we have (7.1).

If an l.c. K-manifold is an l.c. K-space form $\tilde{M}(H)$, then we have

(7.8) $$\tilde{H}_B(X,Z) = \frac{1}{2}(H - \|\langle\alpha, X\rangle\|^2 - \|\langle\alpha, Z\rangle\|^2 - \|\alpha\|^2 + \tilde{\nabla}_X\alpha, X\rangle + \langle\tilde{\nabla}_Z\alpha, Z\rangle).$$

Thus we have from (7.1) and (7.8)

COROLLARY 7.2. *Let $\tilde{M}(H)$ be an l.c. K-space form and M be a mixed foliate submanifold of $\tilde{M}(H)$. Then we have*

(7.9) $$H - \|\langle\alpha, X\rangle\|^2 - \|\langle\alpha, Z\rangle\|^2 - \|\alpha\|^2 + \langle\tilde{\nabla}_X\alpha, X\rangle + \langle\tilde{\nabla}_Z\alpha, Z\rangle - 4\|\langle\beta, Z\rangle\|^2 = -2\|A_{JZ}X\|^2 - 2\|A_{JZ}JX\|^2$$

for any units X in D and Z in $D^\perp$.

References

1. A. Bejancu, *CR-submanifolds of a Kaehler manifold* I, Proc. Amer. Math. Soc. **69** (1978), 135-142.
2. ______, *CR-submanifolds of a Kaehler manifolds* II, Trans. Amer. Math. Soc. **250** (1979), 333-345.
3. D.E. Blair and B.Y. Chen, *On CR-submanifolds of Hermitian manifolds,* Israel J. Math. **34** (1979), 353-363.
4. B.Y. Chen, *CR-submanifolds of a Kaehlerian manifold* I, J. Differential Geometry, **16** (1981), 305-322.
5. ______, *CR-submanifolds of a Kaehlerian manifold* II, J. Differential Geometry, **16** (1981), 493-509.

6. _______, *Geometry of Submanifolds*, M. Dekker, New York, 1973.
7. T. Kashiwada, *Some properties of locally conformal Kähler manifolds*, to appear.
8. _______, *On V-Killng forms in a locally conformal Kähler manifold with parallel Lee form*, to appear.
9. _______, *On V-harmonic form in compact conformal Kähler manifolds*, to appear.
10. K. Matsumoto, *On submanifolds of locally conformal Kähler manifolds*, to appear.
11. S. Tachibana, *On the Bochner curvature tensor*, Natur. Sci. Rep. Ochanomizu Univ. **18** (1967), 15–19.
12. K. Yano and M. Kon, *Differential geometry of CR-submanifolds*, Geometriae Dedicata, **10** (1981), 369–391.
13. _______, *Generic submanifolds*, Ann. of Math. **123** (1980), 59–92.
14. L. Vaisman; *On locally conformal almost Kähler manifolds*, Israel J. Math. **24** (1976), 15–19.

Department of Mathematics
Faculty of Education
Yamagata University
Yamagata, Japan