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On f-Best Approximation in Topological Vector Spaces

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For a non-empty subset K of a vector space X, the notion of best approximation in K relative to a real valued function f on X was given by Breckner and Brosowski [1]. Taking X to be a Hausdorff locally convex topological vector space and f to be a continuous sublinear functional on X, certain results on best approximation relative to the functional f were proved in [1], [3] and [6]. Here we give some characterization of *f-best approximants* in Hausdorff locally convex topological vector spaces X and discuss some other notions in the theory of best approximation relative to a functional f on X.

Let X be a Hausdorff locally convex topological vector space, f a real continuous sublinear functional on X and K a non-empty closed subspace of X. For a given $x\in X$, an element $k^*\in K$ is said to be *f-best approximation to x* in K if

$$f(x-k^*)=inf\{f(x-k) : k\in K\}\equiv f_K(x) \text{ or } f(x-K).$$

The following proposition characterizes *f-best approximation* elements when f is a symmetric (i.e. $f(-x)=f(x)$ and so $f(\alpha x)=|\alpha|f(x)$ for every scalar α) sublinear functional on X.

Proposition 1. Let $x\in X$. *Then* $k_0\in L_{K,f}(x)\equiv\{k^*\in K : f(x-k^*)=f_K(x)\}$ *if and only if* $k_0\in L_{K,f}[tx+(1-t)k_0]$ *for every scalar t.*

Proof. Let $k_0\in L_{K,f}(x)$. Then for all $k\in K$

$$\begin{aligned} f[tx+(1-t)k_0-k] &= |t|f\left[x-k_0+\frac{k_0-k}{t}\right],\ t\neq 0 \\ &\geqslant |t|f(x-k_0) \\ &= f[tx+(1-t)k_0-k_0]. \end{aligned}$$

For $t=0$, $f[tx+(1-t)k_0-k]\geqslant f[tx+(1-t)k_0-k_0]$ is obvious. Hence $k_0\in L_{K,f}[tx+(1-t)k_0]$.

Conversely, let $k_0\in L_{K,f}[tx+(1-t)k_0]$ for every scalar t. Then for $t=1$, $k_0\in L_{K,f}(x)$.

In order to give another characterization of *f-best approximation* elements, we extend to spaces X the notion of orthogonality in normed linear spaces.

For $x, y\in X$, x is said to be *f-orthogonal* to y, written as $x\perp_f y$ if

$$f(x)\leqslant f(x+\alpha y)$$

for every scalar α.

x is said to be *f-orthogonal* to a set $K\subset X$, $x\perp_f K$, if

$$x\perp_f y \text{ for all } y\in K.$$

Proposition 2. *Let f is a symmetric sublinear functional on X. Then* $k_0\in K$ *is f-best approximation*

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to $x \in X$ *if and only if* $(x-k_0)\perp_f K$.

Proof of this proposition is similar to that of Lemma 1.14 [7].

Remark. $L_{K,f}(x)$ *is empty for every* $x \in X$ *if there exists no* $y \in X/\{0\}$ *such that* $y \perp_f K$.

Now we introduce the notion of *f-coapproximation* in the space X.

An element $k_0 \in K$ is said to be *f-coapproximation* to $x \in X$ by elements of K if

$$f(k_0-k) \leqslant f(x-k)$$

for all $k \in K$.

We shall denote by $R_{K,f}(x)$, the set of all *f-coapproximations* to x in K.

The following proposition gives a relation between *f-best approximation* and *f-coapproximation* for symmetric sublinear functional on X.

Proposition 3. $R_{K,f}(x) = \{k_0 \in K : k_0 \in \bigcap_{k \in K} L_{\langle k_0, x\rangle, f}(k)\}$, *where* $\langle k_0, x\rangle = \{\alpha x + (1-\alpha)\, k_0 : \alpha \text{ scalar}\}$ *is the linear manifold spanned by* k_0 *and* x.

Its proof is similar to that of Proposition 2.1 [2].

The following propositions characterize *f-coapproximation* elements for symmetric sublinear functional f on X:

Proposition 4. $k_0 \in K$ *is f-coapproximation to* $x \in X$ *if and only if* $K \perp_f (x-k_0)$.

Proof. $K \perp_f (x-k_0) \langle==\rangle f[k+\alpha(x-k_0)] \geqslant f(k)$, α scalar, $k \in K$

$\langle==\rangle |\alpha| f[\alpha^{-1}k + x - k_0] \geqslant f(k)$, $\alpha \neq 0$, $k \in K$

$\langle==\rangle f[x-k_0+k'] \geqslant f(k')$, $k' \in K$

$\langle==\rangle f(x-k'') \geqslant f(k_0-k'')$, $k'' \in K$

$\langle==\rangle k_0 \in R_{K,f}(x)$.

Proposition 5. $k_0 \in R_{K,f}(x)$ *if and only if* $k_0 \in R_{K,f}[tx+(1-t)k_0]$ *for all scalars* t.

Proposition 6. $k_0 \in R_{K,f}(x)$ *if and only if for all* $k \in K$ *and* $|t| \geqslant 1$, $f[x-k_0+t(k_0-k)] \geqslant f(k_0-k)$.

Proposition 7. $k_0 \in R_{K,f}(x)$ *if and only if for all* $k \in K$, $(1-t)k_0 + tk \in R_{K,f}(x)$, $0 \leqslant t \leqslant 1$.

Proposition 8. $k_0 \in R_{K,f}(x)$ *implies* $\alpha k_0 + \beta k \in R_{K,f}(\alpha x + \beta k)$ *for all* $k \in K$, $0 \neq \alpha, \beta$ *scalars.*

The proofs of these propositions can be developed on the lines of Propositions 2.2, 2.3, 2.4 and 2.5 respectively of [2].

Next we introduce the notion of strong *f-approximation* in the space X.

An element $k_0 \in K$ is said to be a strong *f-approximation* of x by elements of K if there exists an $r > 0$ $(r \leqslant 1)$ such that

$$f(x-k_0) + rf(k_0-k) \leqslant f(x-k)$$

for all $k \in K$.

We shall denote the collection of all such $k_0 \in K$ by $L_{S,K,f}(x)$. Clearly, strong *f-approximation* element is an *f-best approximation* element.

The following propositions characterize strong *f-approximation* elements for symmetric f:

Proposition 9. $k_0 \in L_{S,K,f}(x)$ *if and only if for all* $k \in K$, $f[tx+(1-t)k_0-k] \geqslant f(x-k_0) +$

$rf(k_0-k)$, $|t|\geqslant 1$.

Proposition 10. *$k_0 \in L_{S,K,f}(x)$ if and only if $k_0 \in L_{S,K,f}[tx+(1-t)k_0]$ for all scalars t.*

Proposition 11. *$k_0 \in L_{S,K,f}(x)$ if and only if for all $k \in K$, $f[x-k_0(1-t)-tk] \geqslant f(x-k_0)+rf(k_0-k)$ for all scalars t.*

Proposition 12. *$k_0 \in L_{S,K,f}(x)$ if and only if $\alpha k_0+\beta k \in L_{S,K,f}(\alpha x+\beta k)$, $\alpha \neq 0$, β scalars, and $k \in K$.*

The proof of these propositions can be developed as of propositions 3.1, 3.2, 3.3 and 3.4 respectively of [2].

Next we consider strong *f-coapproximation* elements in the space X.

An element $k_0 \in K$ is said to be a ***strong f-coapproximation*** of x by elements of K if there exists $r>0 (r \leqslant 1)$ such that

$$f(x-k) \geqslant f(k_0-k)+rf(x-k_0)$$

for all $k \in K$. We shall denote the collection of all such k_0 by $R_{S,K,f}(x)$.

Strong *f-coapproximation* element is an *f-coapproximation.*

Propositions 9, 10, 11 and 12 hold for strong *f-coapproximation* (for symmetric f). Proposition 9 holds for all scalars and Proposition 11 holds only for $|t| \geqslant 1$.

The following proposition whose proof is similar to that of 4.1 [2], gives a relation between strong *f-approximation* and *strong f-coapproximation* for symmetric f.

Proposition 13. *$R_{S,K,f}(x)=\{k_0 \in K : k_0 \in \bigcap_{k \in K} L_{S,\langle k_0,x\rangle f,}(k)\}$, where $\langle k_0, x\rangle$ is the linear manifold generated by k_0 and x.*

Next we discuss (ε)-*f-approximation* in spaces X for a given $\varepsilon>0$.

An element $k_0 \in K$ is said te be (ε)-***f-approximation*** to $x \in X$ if

$$f(x-k_0) \leqslant f(x-K)+\varepsilon.$$

The set of all such k_0 is denoted by $L_{K,f}(x, \varepsilon)$.

The following proposition characterizes the set $L_{K,f}(x,\varepsilon)$ for a symmetric f.

Proposition 14. *$k_0 \in L_{K,f}(x,\varepsilon)$ if and only if $k_0 \in L_{K,f}[tx+(1-t)k_0, \varepsilon]$ for all scalars t with $|t| \leqslant 1$.*

Its proof is similar to that of Theorem 3.3 [5].

Finally, we introduce simultaneous *f-approximation* elements in the space X.

An element $k_0 \in K$ is said to be a ***simultaneous f-approximation*** of the pair $x_1, x_2 \in X$ if

$$Max\ \{f(x_1-k_0),\ f(x_2-k_0)\} = \inf_{k \in K} Max\{f(x_1-k),\ f(x_2-k)\}.$$

The following proposition whose proof is similar to that of Theorem 3.1 [4], gives a relationship between elements of simultaneous *f-approximation* and *f-best approximation.*

Proposition 15. *Every pair $x_1, x_2 \in K^{\perp} \equiv \{y \in X : y \perp_f K\}$ has a simultaneous f-approximation in K which is also an f-best approximation of the arithemetic mean of x_1, x_2 if x_1, x_2 are linearly dependent and f-orthogonality in X is f-homogeneous i.e. $x \perp_f K$ implies $\alpha x \perp_f K$ for every scalar α.*

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