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Divisibility Property of Integers Formed by Repeated Blocks of Digits

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Let c be an integer such that in position representation base $b>1$ the "digits" $0,1,2,\ldots,b-1$ form a finite sequence of a block of these integers. For example, the number 134, 134, 134 written in a base $b>4$ repeats the block 134 three times. It is noted that if the block is a palindrome, then so is any number which is a finite sequence of the block. The question we address is the following: If n is a given integer and $k=a_1a_2\ldots a_r$ is a given block of digits in base b, then does there exist a number expressible as a finite sequence of this block in base b positional notation which is divisible by n? The given theorem and corollaries consider divisibility of this from.

Therem. *Let n be a given integer and $k=a_1a_2\ldots a_r$, $r \in Z^+$, be a given block of digits in base $b>1$.* If $(n,b)=1$, *then there exists an integer m expressible as a finite sequence of block k which is a multiple of n.*

Proof. We will adopt the notational abbreviation of $\overbrace{k\ldots k}^{s}$ to express the number $a_1a_2\ldots a_ra_1a_2\ldots a_r\ldots a_1a_2\ldots a_r$ written in base b positional notation with block k repeated s times. Consider the following sets of integers written in base b:

$S=\{k,kk,\ldots,\overbrace{k\ldots k}^{n+1}\}$ and $S_i=\{x\varepsilon S\,|\,x\equiv i\ (mod\ n)\}$, $0\leq i\leq n-1$. Since $n+1=\#(S)>n=\#\{S_i\,|\,0\leq i\leq n-1\}$, by Pigeonhole principle we have that their exists two distinct elements $\overbrace{k\ldots k}^{r}$ and $\overbrace{k\ldots k}^{s}$, with $r>s$, in some S_i which are thus congruent modulo n. Hence, $n\,|\,((\overbrace{k\ldots k}^{r})-(\overbrace{k\ldots k}^{s}))=\overbrace{k\ldots k}^{r-s}\ \overbrace{o\ldots o}^{s}=(\overbrace{k\ldots k}^{r-s})\times b^s$.

Since $(n,b^s)=1$, we have $n\,|\,\overbrace{k\ldots k}^{r-s}$—completing the proof.

Corollary 1. *Let block $k=a_1a_2\ldots a_r$ be given as above. If $(n,b)=1$, then n divides an integer expressible as n or fewer copies of the given block in base b positional notation.*

Proof. In the proof of the theorem, $1\leq r,s\leq n+1$, and so $r-s\leq(n+1)-1=n$.

Corollary 2. *If any block q of digits base $b>1$ is given and if n divides none of the numbers of* *the set* $\{q, qq, \ldots, \overbrace{q\ldots q}^{n}\}$, *then* $(n, b) > 1$.

Proof. Contraposition of the theorem.

Corollary 3. *Let n and k be given as in the theorem. If* $t < n$ *and* $(t, b) = 1$, *then there exists an integer expressible as a finite sequence of block k in positional notation base b which is a multiple of t.*

Proof. Apply the Pigeonhole Principle as in the proof of the theorem.

Corollary 4. *If n is an odd iteger and* $5 \nmid n$, *then n is a divisor of some integer expressible as a finite sequence of block* $k = d_1 d_2 \ldots d_v$, $v \in Z^+$, $0 \leq d_i \leq 9$ *for each i, in base 10 positional notation.*

Proof. Let $b = 10$. Then by the hypothesis on n, $(n, 10) = 1$ and so the result follows from the theorem.

An example is offered to demonstrate this curious result. Consider block 1273 in base 10 and integer 9. Since $(9, 10) = 1$ we know, by the theorem or corollary 4, that 9 divides a finite sequence of this block. You may show that, in fact, 9|127, 312, 731, 273.

The given results may be considered as special divisibility criteria for integers which include, as particular cases, certain palindromic multiples.