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A Note on Generalized Inverses in Regular Near-ring

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Introduction

The concepts of a regular near-ring was introduced in 1968 by J.C. Beidlemann [1] and several elementary properties of such near-rings were developed. Later St eve Ligh [4] was the first to give some structure theory for regular near-rings. He obtained some characterizations of regular near-rings and some results. In 1976, Steve Ligh and Yuzo Utumi [5] introduced the concepts of strongly regular near-rings and obtained the relating properties.

In this paper, we investigated the generalized inverses in regular near-rings.

Definitions

Throughout this paper, N will mean a right near-ring. An element a in N is regular if $a=axa$ has a solution in N and any such solution x is called a generalized inverse of a. As the theory of rings [3] the element a in N will be called unit regular if N has an identity and a has an invertible generalized inverse. A reflexive inverse of a is a near-ring element x such that $a=axa$ and $x=xax$. Every regular element possesses a reflexive inverse for $a=axa$ implies that $y=xax$ is a reflexive inverse of a. The near-ring N is regular (respectively unit regular) if each of its elements is regular (respectively unit regular). Finally N is strongly regular if for each a in N there is an element x in N such that $a=a^2x$. Equivalently, for each a there is an element x with $x=ax^2$.

Strongly regularity always implies regularity and, in the presence of an identity element, strongly regularity implies unit regularity which implies regularity. A near-ring N is called a near-field if and only if the set of nonzero elements forms a multiplicative group. The basic reference for near-ring concepts is [7]. This paper proceeds in the spirit of Bitzer's method of proof [2].

Main results

Here we collect some results of J.C. Beidleman and Steve Ligh.

Theorem 1 [1] *A near-ring N is regular if and only if for each nonzero element a of N there exists an idempotent b such that $Na=Nb$.*

Theorm 2 [1] *If N is a regular near-ring with identity e and if N has no nonzero divisors of zero, then N is a near-field.*

Proof. Let a be a non-zero element of N. Then there exists an element x of N such that $axa=a$. Since N contains no non-zero divisors of zero it follows that $xa=e$. Similarly, there exists an element y such that $yx=e$, and therefore $ax=(yx)(ax)=y(xa)x=yx=e$. Hence N is a near-field.

Theorem 3 [1] *Every near-ring with identity is isomorphic to a sup-near-ring of a regular near-ring.*

Theorem 4 [4] *A regular near-ring N is a near-field if and only if the nonzero idempotent of N is the identity.*

Theorem 5. *For a nonzero distributive regular element a of a near-ring N, the following statements are equivalent;*

(i) a has an unique generalized inverse,

(ii) a is neither a right nor a left divisor of zero,

(iii) N has an identity and a is an unit element.

Proof. (i)→(ii)

If x is the unique generalized inverse of a and if $ab=0$ or $ba=0$, then $a(x+b)a=a$. By uniqueness, $x+b=x$, whence $b=0$.

(ii)→(iii)

Suppose a is neither a right nor a left divisor of zero. Choose an x with $a=axa$. For any b in N we have $a(b-xab)=0=(b-bax)a$ and therefore, $xab=b=bax$. Thus xa is a left identity and ax is a right identity for N. Hence $e=ax=xa$ is the identity for N and a is clearly an unit element.

(iii)→(i)

If N has the identity e and a is an unit element, then $a=axa$ implies $ax=e=xa$, so $x=$a.

From the theorem 5, we obtain the following theorem.

Theorem 6. *A nonzero near-ring N with identity is a near-field if and only if each nonzero distributive element of N has an unique generalized inverse.*

Theorem 7. *If a is a regular element of the distributive near-ring N, then the following statements are equivalent;*

(i) a has an unique reflexive inverse,

(ii) there is an element x in N such that $a=axa$ and both ax and xa are central idempotents.

(iii) if $a=aya$, then $ay=ya$,

(iv) if $a=aya=axa$, then $ay=ax=xa=ya$.

Proof. (i)→(ii).

Let x be the unique element of N for which $a=axa$ and $x=xax$. For any y in N the elements $x+y-xay$ and $x+y-yax$ are generalized inverses of a and hence,

$$x=(x+y-xay)a(x+y-xay)=x+yax-xayax$$
$$=(x+y-yax)a(x+y-yax)=x+xay-xayax.$$

Therefore, $yax=xay$ for every y in N. Letting y be ax and xa successively, we have $ax=(ax)^2=xa^2x=(xa)^2=xa$, since $(ax)ax=(xa)ax$ implies $ax=xa^2x$ and $(xa)ax=(xa)xa$ implies $xa^2x=xa$.

(ii)→(iii).

Choose an element x with $a=axa$ and both ax and xa in the center.

Then $ax=axax=a(xa)x=(ax)(xa)=(xa)(ax)=x(ax)a$.

Hence, if $a=aya$, then $ay=(ax)(ay)=(ay)(ax)=(aya)x=ax=xa$

$$=x(aya)=(ya)(xa)=ya.$$

(iii)→(iv).

If $a=aya=axa$, then by (iii), $ay=ya=y(axa)=(ya)(xa)=(ay)(ax)(aya)x=ax=xa$.

(iv)→(i).

If y and x are reflexive inverses of a, then $y=yay=yax=xax=x$.

If every idempotent of N is central, regularity implies strongly regularity. For $a=axa=(ax)a=a(ax)=a^2x$.

Lemma 8. *A near-ring N is strongly regular if and only if N is regular and every idempotent of N is central.*

Therefore the next result is an immediate consequence of Theorem 7 and Lemma 8.

Theorem 9. *A near-ring N is strongly regular if and only if every element of N has an unique reflexive inverse.*

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