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On the Conditions for One-sided Inverses to be Two-sided in Group Algebras

by Kyoung Hee Lee
Seoul National University, Seoul, Korea

1. **Introduction.** This paper contains some remarks about the Kaplansky's conjecture [1]: In a group algebra, are one-sided inverses two-sided? Hilbert space methods in the discrete group algebra have proved this to be true in characteristic 0; see [3]. But the characteristic p case remains open.

G. Losey showed this property (that one-sided inverses are two-sided) holds for group algebras of particular groups over arbitrary fields such that finite groups, abelian groups, nilpotent groups, and free groups [2].

Here we examine whether or not the conjecture is true in group algebras of supersolvable groups and solvable groups.

2. **Preliminaries.** A group G is supersolvable if it has a normal series with cyclic factors: $G=G_0\supseteq G_1\supseteq G_2\supseteq G_3\supseteq\dots\supseteq G_n=\{1\}$.

Theorem 1. [4] *Let RG be a group ring of a supersolvable group G over a commutative ring with identity R. Suppose that R has no nontrivial idempotents and that if G has an element of prime order p then $p\notin UR$. Then RG has no nontrivial idempotents.*

A group G is an $\mathcal{F}$-group if RG is strongly finite for all strongly finite rings R (A ring R is strongly finite means that for any pair A, B of $n\times n$ matrices over R, $AB=I_n$ implies $BA=I_n$ for all positive intergers n). We know that G is an $\mathcal{F}$-group if and only if RG is 1-finite, i.e., each right invertible element is left invertible, since $(RG)_n=(R_n)G$.

Lemma 1. *A group G is an $\mathcal{F}$-group if and only if it is locally an $\mathcal{F}$-group.*

Theorem 2. [2] *Let G be a group and H a subgroup of G of finite index. If H is an $\mathcal{F}$-group, then G is an $\mathcal{F}$-group.*

3. **Some Results.** First we give a result for supersolvable groups:

Theorem 3. *Let G be a supersolvable group and K be a field of arbitrary characteristic. Then each element which is right-invertible is left-invertible in the group algebra NG.*

Proof. Let $ab=1$ for $a, b\in KG$. We set $e=ba$. Then e is an idempotent, since $e^2=(ba)(ba)=b(ab)a=ba=e$. Hence e is trival by Theorem 1, i.e., $e=0$ or 1. But if $e=0$, then $1=(ab)(ab)=a(ba)b=0$, a contradiction. Therefore, $ba=1$.

Remark. In fact, this theorem remains true for a group ring over a commutative ring R with identity. ([2], Theorem 1)

Theorem 4. *A solvable torsion group G is an $\mathcal{F}$-group.*

Proof. There exists a normal sequence $G=G_0\supseteq G_1\supseteq\ldots\supseteq G_n=\{1\}$, where G_i/G_{i+1} is ablelian for each i. Since g is a torsion group, G_i/G_{i+1} is an abelian torsion group, so it is locally finite for all i. Hence G is also locally finite. By Lemma 1 and Theorem 2, G is an $\mathcal{F}$-group.

Now we investigate our assertion for solvable groups. But in this direction, there are two short theorems:

Theorem 5. [2] *Let G be a group, N a normal subgroup and assume (a) G/N is abelian, (b) N is finite. Then G is an $\mathcal{F}$-group,*

Proof. It is sufficient to assume G/N finitely generated. Then $G/N\cong G_1/N\times\ldots\times G_k/N$, where each factor G_i/N is cyclic. If each G_i/N is finite cyclic, then G_i/N is an $\mathcal{F}$-group by Theorem 2, and hence so is G/N. For the case that G_i/N is infinite cyclic, let xN be a generator of N. Then $y\to x^{-1}yx$ is an automorphism of N. Since N is finite, N has a finite automorphism group, so x^m centralizes N *for* some $m>0$. Thus $N^*=(x^m, N)=(x^m)\times N$ is an $\mathcal{F}$-group and $[G:N]=m$. By Theorem 2, G is an $\mathcal{F}$-group.

Theorem 6. [2] *Let G be a group, N a normal subgroup and assume (a) G/N is abelian, (b) N is finitely generated abelian. Then G is an $\mathcal{F}$-group.*

From this, we obtain the following corollaries:

Let $C(G)$ be the center of a group G.

Corollary. 1. If $C(G)$ has a finite index in G, then G is an $\mathcal{F}$-group.

Proof. By Schur's lemma [4], the commutator subgroup G' is finite. And G/G' is abelian. So the result follows by Theorem 5.

A group in which each element has a finite number of congugates is called an FC-group.

Corollary 2. A finitely generated FC-group is an $\mathcal{F}$-group.

Proof. Let $g_1, g_2, \ldots, g_n$ be generators of G. Then $[G:C(G)]<\infty$ for all i and consequently $[G:C(G)]=[G:{}_{i=1}\cap^n C(g_i)]<\infty$. Thus G is an $\mathcal{F}$-group by Corollary 1.

References

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