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On S-Ideals in BCK-Algebras

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I. Preliminaries

Lex X be a non-empty set with a binary operation $*$ and suppose there is a constant 0 in X. Then $(X, *, 0)$ (or simply denoted by X) is called a *BCK-algebra* if the following conditions hold:

(1) $(x*y)*(x*z)\leq z*y$,

(2) $x*(x*y)\leq y$,

(3) $x\leq x$,

(4) $0\leq x$,

(5) $x\leq y$ *and* $y\leq x$ imply $x=y$.

where $x\leq y$ means $x*y=0$.

Example 1. ([4]) Let X be the set of all natural numbers: $0, 1, 2, \ldots$. For two natural numbers x, y, we put

$$x*y=\begin{cases} 0, & \text{if } x\leq y, \\ x-y, & \text{if } y<x. \end{cases}$$

Then X is a BCK-algebra.

Example 2. ([4]) Let $X=\{0, 1, 2, \ldots\}$ with the natural order. Then we define

$$x*y=\begin{cases} 0, & \text{if } x\leq y, \\ 1, & \text{if } y<x \text{ and } y\neq 0, \\ x, & \text{if } y<x \text{ and } y=0. \end{cases}$$

Under the definition of $*, X$ is a BCK-algebra.

Let X be a BCK-algebra and suppose that there is an element 1 in X such that $x\leq 1$ for all $x\in X$. Then X is called a *bounded* BCK-algebra, and 1 is said to be a *unit* of X.

We shall state some properties on BCK-algebra:

(1) $x\leq y$ implies $z*y\leq z*x$

(2) $x\leq y$, $y\leq z$ implies $x\leq z$

(3) $(x*y)*z\leq (z*x)*y$

(4) $(x*y)*z=(x*z)*y$

(5) $(x*y)\leq z$ implies $x*z\leq y$

(6) $(x*y)*(z*y)\leq x*z$

(7) $x\leq y$ implies $x*z\leq y*z$

(8) $x*y\leq x$

(9) $x*0=x$

for all x, y, z in BCK-algebra X.

Theorem 1. ([2]) *Let $X=(X, *, 0)$ be a BCK-algebra without unit, and let $1 \notin X$. we define, for all $x \in X$,*

$$x*1=0 \text{ and } 1*1=0.$$

Next we define

$$1*x=1$$

for every $x \in X$.

Then the algebra X with 1 (denoted by X') is a BCK-algebra.

For BCK-algebras X and Y, a mapping $f: X \rightarrow Y$ is called a *homomorphism* if for any $x, y \in X$, $f(x*y)=f(x)*f(y)$. If a homomorphism $f: X \rightarrow Y$ is onto, f is called an *epimorphism*. A non-empty subset A of a BCK-algebra X is called an *ideal* if the following conditions are satisfied:

(1) $0 \in A$,

(2) $x \in A$ and $y*x \in A$ imply $y \in A$.

Theorem 2. ([2]) *Let $f: X \rightarrow Y$ be a homomorphism. Then the kernel of f, $Ker(f)$, is an ideal of X.*

II. S-Ideals in BCK-algebras

In the theorem 1, the algebra X is an ideal of the algebra X' obtained by adding 1. Then $x*1=0 \in X$ but $1*x=1 \notin X$. In general it does not hold in the ideal A of BCK-algebra that $x*y \in A$ imply $y*x \in A$. Hence we define a *symmetric ideal* (briefly *S-ideal*) in BCK-algebra.

Let A be a non-empty subset of BCK-algebra X. A is called to be a *S-ideal* if

(1) $0 \in A$,

(2) $x \in A$ and $x*y \in A$ imply $y \in A$.

By definition, the simplest examples of S-ideals are $\{0\}$ and X.

Theorem 3. *Let $f: X \rightarrow Y$ be a homomorphism of BCK-algebras. Then the kernel of f, $Ker(f)$, is a S-ideal of X.*

Proof. Since $f(0)=0, 0 \in Ker(f)$, Let $x \in Ker(f)$, $x*y \in Ker(f)$, then $f(x)=0 \in \{0\}$, $f(x)*f(y)=f(x*y)=0 \in \{0\}$. Since $\{0\}$ is a S-ideal, $f(y) \in \{0\}$. Hence $y \in Ker(f)$. Therefore $Ker(f)$ is a S-ideal of X.

Theorem 4. *If x is element of S-ideal A and $x \leq y$, then $y \in A$.*

Proof. $x \leq y$ imply $x*y=0 \in A$. Since $x \in A$ and since A is a S-ideal, $y \in A$.

We set $A_t=\{x \in X \mid x \leq t \text{ for fixed } t \in X\}$. A_t is not a S-ideal. For example, fix $t=5$ in example 1, and the set A_5 is not a S-ideal.

Theorem 5. *Let X be a BCK-algebra. Then any set A_t is a S-ideal if and only if $x \leq z$ and $x*y \leq z$ imply $y \leq z$ for any $x, y, z \in X$.*

Proof. ($\Rightarrow$) Suppose that any A_t is a S-ideal. Let $x \leq z$ and $x*y \leq z$. By the hypothesis, A_z is a S-ideal. Hence $x \in A_z$ and $x*y \in A_z$ imply $y \in A_z$. Therefore $y \leq z$.

$\Leftarrow$) Consider A_z for any $z \in X$. Obviously $0 \in A_z$.

Let $x \in A_z$ and $x * y \in A_z$. Then $x \leq z$ and $x * y \leq z$. By the hypothesis, $y \leq z$ and hence $y \in A_z$.

Remark. In general, Any ideal is not always S-ideal. For example, in the theorem 1, the algebra X is an ideal of X', but X is not S-ideal of X'.

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