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# Best Simultaneous Approximation in Metric Spaces

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Let $(X, d)$ be a metric space, $K$ a subset of $X$, and $x_1, x_2 \in X$. An element $k_0 \in K$ is called ***a best simultaneous approximation*** (*b.s.a.*) to $x_1$ and $x_2$ if

$$max\ \{d(x_1, k_0),\ d(x_2, k_0)\} = \inf_{k\in K} max\ \{d(x_1, k),\ d(x_2, k)\}.$$

The problem of *b.s.a.* has been studied by a battery of mathematicians (cf. [2]). As remarked by F. Deutsch [MR #13556, 51(1976)] the problem of *b.s.a.* can be viewed as the problem of ordinary best approximation in a certain product space. Specifically, if $Y$ denotes the product space $X\times X$ equipped with the metric $d^*$,

$$d^*\{(x_1, x_2), (y_1, y_2)\} = Max\ \{d(x_1, y_1),\ d(x_2, y_2)\}$$

and $D(K) = \{(k, k): k\in K\}$, then as shown below, $k_0 \in K$ is a *b.s.a.* to $x_1$ and $x_2$ iff $(k_0, k_0) \in D(K)$ is a best approximation to $(x_1, x_2) \in Y$:

$$\begin{aligned} & (k_0, k_0) \text{ is a best approximation to } (x_1, x_2) \in Y \\ \iff & d^*\{(x_1, x_2), (k_0, k_0)\} = \inf_{k\in K} [d^*\{(x_1, x_2),\ (k, k)\}] \\ \iff & Max\ \{d(x_1, k_0),\ d(x_2, k_0)\} = \inf_{k\in K} [Max\{d(x_1, k), d(x_2, k)\} \\ \iff & k_0 \text{ is a } b.s.a. \text{ to } x_1 \text{ and } x_2. \end{aligned}$$

Therefore if $inf\ [d^*\{(x_1, x_2),\ (k, k)\} : (k, k) \in D(K)]$ is attained at some $(k_0, k_0) \in D(K)$ then the problem of *b.s.a.* has a solution i.e. if $D(K)$ is proximinal in $(X\times X,\ d^*)$ then the problem of *b.s.a.* has a solution and if it is Chebyshev then the problem has a unique solution (The various conditions under which a subset of a metric space is proximinal of Chebyshev have been discussed in [1]).

Viewed in this way, the results already proved by direct methods (cf. [2]) are much more easily obtained and even more general results can be proved. In fact the whole problem can be extended further, e.g., by considering different metrics on the product space and also by taking the product of $X$ with itself $n$ times, $n \geqslant 2$.

### Uniqueness of b.s.a. in Metric Space:

**Theorem 1.** *If $K$ is a convex proximinal set in a strictly convex metric space $(X, d)$ then there exists a unique b.s.a. in $K$ to $x_1, x_2$ of $X$ (For the various concepts in the statement we refer to* [1]).

**Proof.** Let, if possible, $k_1^*$, $k_2^*$ in $K$ be two b.s.a. to the pair $x_1, x_2$ of $X$ i.e.

$$Max\{d(k_1^*, x_1),\ d(k_1^*, x_2)\} = r = Max\{d(k_2^*, x_1),\ d(k_2^*, x_2)\},$$

where $r = inf[Max\{d(k, x_1),\ d(k, x_2)\} :\ k \in K]$.

Then

$d(k_1^*, x_1) \leqslant r, d(k_2^*, x_1) \leqslant r,\ d(k_1^*, x_2) \leqslant r$ and $d(k_2^*, x_2) \leqslant r$. Since $X$ is strictly convex, we have

$d(k^*, x_1) < r$, $d(k^*, x_2) < r$ unless $k_1^* = k_2^*$, where $k^* \in K$ is the unique mid point of $k_1^*$, $k_2^*$. The existence of $k^* \in K$ is guaranteed by the strict convexity of $X$ and the convexity of $K$ (cf. [1]). This means that

$$Max\{d(k^*, x_1),\ d(k^*, x_2)\} < r \text{ unless } k_1^* = k_2^*.$$

But this contradicts the definition of $r$ since $k^* \in K$. Hence $k_1^* = k_2^*$.

Combining the above theorem and remarks preceding it we have the following:

**Theorem** 2. *If $K$ is a conovex set in a strictly convex metrie space $(X, d)$ such that $D(K)$ is proximinal in $(X \times X,\ d^*)$ then b.s.a. for each pair $x_1, x_2$ in $X$ exists in $K$ and is unique.*

**Note.** The above two theorems can be extended to finding b.s.a. of $n$ elements of $X$.

## References

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