韓國數學敎育學會誌<數學敎育>
1984. 6. Vol. XXII, No. 3.

Journal of the Korea Society of Mathematical Education
Jun. 1984. Vol. XXII, No. 3.

Lagrange Stability and Poisson Stability in Transformation Groups

by Jong Suh Park and Yun Hoy Gu

1. Introduction

Let (X,T) be a transformation group. A point x of X is said to be *Lagrange stable* if the orbit closure $\overline{xT}$ of x is compact. Suppose that the phase group T is noncompact. Given a point x of X, the limit set $L(x)$ of x is defined to be the intersection of $\overline{xK^c}$ for all compact subsets K of T. A point x of X is said to be Poisson stable if x belongs to $L(x)$. In this paper some properties of the above concepts are discussed. These results are the generalization of those in flows.

2. Preliminaries

Definition 2.1. A transformation group is a triple (X,T,ψ) where X is a topological space called the phase space, T is a topological group called the phase group and $\psi: X\times T\to X, (x,t)\to xt$ is a continuous map such that for all $x\in X$, $s,t\in T$, $xe=x$, $(xs)t=x(st)$ where e is the identity of T.

It will be convenient to suppress the map ψ and then denote the transformation group (X,T,ψ) simply as (X,T).

Definition 2.2. Let (X,T) be a transformation group. T is said to act freely on X if for each point x of X the map $\psi_X: T\to X$, $t\to xt$ is injective.

Definition 2.3. Let (X,T) be a transformation group. A subset M of X is said to be invariant if Mt is contained in M for all $t\in T$.

Definition 2.4. Let (X,T) and (Y,T) be transformation groups. A continuous map $f: X\to Y$ is called a homomorphism if $f(xt)=f(x)t$ for all $x\in X$, $t\in T$.

Definition 2.5. A topological space X is said to be hemicompact if there exists a sequence (Kn) of compact subsets of X such that if K is any compact subset of X, then K is contained in Kn_o for some n_o.

In this paper we only deal with a transformation group whose phase space is Hausdorff and phase group is noncompact.

3. Limit sets

Definition 3.1. Let (X,T) be a transformation group and x a point of X. The limit set $L(x)$ of x is defined by the intersection of $\overline{xK^c}$ for all compact subsets K of T.

For a net (t_α) in T, $t_\alpha\to\infty$ is to mean that (t_α) is ultimately outside each compact subset of T.

Theorem 3.2. *Let (X,T) be a transformation group. For all points x,y of X, $y\in L(x)$ if and* *only if there exists a net* (t_α) *in* T *such that* $t_\alpha \to \infty, xt_\alpha \to y$.

Proof. ($\Rightarrow$) Let (K_α) be the family of all compact subsets of T with the inclusion order and (U_β) the family of all neighborhoods of y with the reverse inclusion order. Define $(\alpha_1, \beta_1) \leq (\alpha_2, \beta_2)$ by $\alpha_1 \leq \alpha_2$ and $\beta_1 \leq \beta_2$. For any (α, β), since $U_\beta \cap xK_\alpha{}^c \neq \phi$, there exists $t_{(\alpha,\beta)} \in K_\alpha{}^c$ such that $xt_{(\alpha,\beta)} \in U_\beta$. It is not hard to show that $t_{(\alpha,\beta)} \to \infty$, $xt_{(\alpha,\beta)} \to y$.

($\Leftarrow$) Given any compact subset K of T, there exists α_0 such that $t_\alpha \in K^c$ for all $\alpha \geq \alpha_0$. For each neighborhood U of y, there exists α_1 such that $xt_\alpha \in U$ for all $\alpha \geq \alpha_1$. There exists α_2 such that α_0, $\alpha_1 \leq \alpha_2$. Clearly $xt_{\alpha 2} \in U \cap xK^c$ and so $U \cap xK^c \neq \phi$. Thus $y \in \overline{xK^c}$, consequently $y \in L(x)$.

Thorem 3.3. *Let* (X,T) *be a transformation group. For any point* x *of* X, $\overline{xT} = xT \cup L(x)$.

Proof. Clearly, $xT \cup L(x) \subset \overline{xT}$. Suppose that $xT \cup L(x) \neq \overline{xT}$. There exists $y \in \overline{xT} - (xT \cup L(x))$. Since $y \notin L(x)$, there exists a compact subset K *of* T such that $y \in \overline{xK^c}$. Thus $U \cap xK^c = \phi$ for some neighborhood U of y. Clealy, xK is a closed subset of X, Since $y \notin xT$, $y \notin xK$. Take $V = U \cap (xK)^c$. Then V is a neighborhood of y and $V \cap xT = \phi$. Thus $y \notin \overline{xT}$. This is a contradiction. Hence $\overline{xT} = xT \cup L(x)$.

Theorem 3.4. *Let* (X,T) *be a transformation group. For any point* x *of* X, $L(x)$ *is invariant.*

Proof. Let y be a point of $L(x)$ and t an element of T. There exists a net (t_α) in T such that $t_\alpha \to \infty$, $xt_\alpha \to y$. It is not hard to show that $t_\alpha t \to \infty$. Since $xt_\alpha t \to yt$, $yt \in L(x)$. Thus $L(x)$ is invariant.

Theorem 3.5. *Let* (X,T) *be a transformation group and* x *a point of* X. *For any point* y *of* xT, $L(y) = L(x)$.

Proof. $y = xt$ for some $t \in T$. Let $z \in L(y)$. Then there exists a net (t_α) in T such that $t_\alpha \to \infty$, $yt_\alpha = xtt_\alpha \to z$. It is not hard to show that $tt_\alpha \to \infty$. Thus $z \in L(x)$. Let $z \in L(x)$. Then there exists a net (t_α) in T such that $t_\alpha \to \infty$, $xt_\alpha = xtt^{-1}t_\alpha = yt^{-1}t_\alpha \to z$. Clearly, $t^{-1}t_\alpha \to \infty$ and so $z \in L(y)$. Hence $L(y) = L(x)$.

Theorem 3.6. *Let* f: $(X,T) \to (Y,T)$ *be a homomorphism of transformation groups. For any point* x *of* X, $f(L(x)) \subset L(f(x))$.

Proof. Let $y \in f(L(x))$. $y = f(z)$ for some $z \in L(x)$. There exists a net (t_α) in T such that $t_\alpha \to \infty$, $xt_\alpha \to z$. $f(xt_\alpha) = f(x)t_\alpha \to f(z) = y$. Thus $y \in L(f(x))$. Hence $f(L(x)) \subset L(f(x))$.

4. Lagrange stability

Definition 4.1. Let (X,T) be a transformation group. A point x of X is said to be *Lagrange stable* if the orbit closure $\overline{xT}$ is compact.

Theorem 4.2. *Let* (X,T) *be a transformation group. If a point* x *of* X *is Lagrange stable, then* $L(x)$ *is nonempty.*

Proof. Let (K_α) be the family of all compact subsets of T. We will prove that $(\overline{xK_\alpha{}^c})$ has the finite intersection property $\bigcap_{i=1}^{n} \overline{xK^c_{\alpha_i}} \supset \overline{\bigcap_{i=1}^{n} xK^c_{\alpha_i}} \supset x\bigcap_{i=1}^{n} K^c_{\alpha_i}$. Since T is noncompact, $\bigcap_{i=1}^{n} K^c_{\alpha_i} \neq \phi$. Thus

$\bigcap_{i=1}^{n} \overline{xK^c_{\alpha_i}} \neq \phi$. Since $\overline{xT}$ is compact, $\bigcap \overline{xK_\alpha^c} = L(x) \neq \phi$.

Corollary 4.3. *Let (X,T) be a transformation group. If a point x of X is Lagrange stable, then $L(x)$ is compact.*

Theorem 4.4. *Let (X,T) be a transformation group. If a point x of X is Lagrange stable, then for each neighborhood U of $L(x)$, there exists a compact subset K of T such that $xK^c \subset U$.*

Proof. By theorem 4.2 $L(x) \neq \phi$. Let (K_α) be the family of all compact subsets of T with the inclusion order. Suppose that there exists a neighborhood U of $L(x)$ such that $xK_\alpha^c \not\subset U$ for all α. For each α, there exists $t_\alpha \in K_\alpha^c$ such that $xt_\alpha \notin U$. (xt_α) is a net in $\overline{xT}$. Since $\overline{xT}$ is compact, (xt_α) has a convergent subnet. Let $xt_\alpha \rightarrow y \in \overline{xT}$. It is not hard to show that $t_\alpha \rightarrow \infty$. Thus $y \in L(x)$ and so U is a neighborhood of y. Since $xt_\alpha \rightarrow y$, there exists α_0 such that $xt_\alpha \in U$ for all $\alpha \geq \alpha_0$. This is a contradiction. Hence the theorem holds.

Theorem 4.5. *Let (X,T) be a transformation group whose phase space is locally compact. Given a point x of X, if $L(x) \neq \phi$ is compact and for each neighborhood U of $L(x)$, there exists a compact subset K of T such that $xK^c \subset U$, then x is Lagrange stable.*

Proof. Since X is locally connected and $L(x)$ is compact, there exists a neighborhood U of $L(x)$ such that $\bar{U}$ is compact. By assumption, there exists a compact subset K of T such that $xK^c \subset U$. $xT = xK \cup xK^c \subset xK \cup U$, $\overline{xT} \subset \overline{xK} \cup U = xK \cup \bar{U}$. $xK \cup \bar{U}$ is compact. Thus $\overline{xT}$ is compact.

Theorem 4.6. *Let (X,T) be a transformation group whose phase spase is locally compact and phase group is hemicompact. There exists a sequence (K_n) of compact subsets of T such that if K is any compact subset of T, then $K \subset Kn_0$ for some n_0. Suppose that Kn^c is connected for all n. Given a point x of X, if $L(x) \neq \phi$ is compact, then x is Lagrange stable.*

Proof. Suppose that there exists a neighborhood U of $L(x)$ such that $xKn^c \not\subset U$ for all n. Since X is locally compact and $L(x)$ is compact, there exists a neighborhood V of $L(x)$ such that $\bar{V} \subset U$ and $\bar{V}$ is compact. For each n, there exists $t_n \in Kn^c$ such that $xt_n \notin U$ so that $xt_n \notin \bar{V}$. Since $L(x) \neq \phi$, there exists $y \in L(x)$. V is a neighborhood of y. For each n, since $y \in \overline{xKn^c}$, $V \cap xKn^c \neq \phi$ so there exists $Sn \in Kn^c$ such that $xSn \in V$. For each n, since Kn^c is connected, there exists $r_n \in Kn^c$ such that $xr_n \in \partial V$. (xr_n) is a sequence in ∂V. Since ∂V is compact, (xr_n) has a convergent subsquence. Let $xr_n \rightarrow z \in \partial V$. It is not hard to show that $r_n \rightarrow \infty$. Since $z \in L(x) \subset V$, $V \cap \partial V \neq \phi$ and this is a contradiction. Thus for any neighborhood U of $L(x)$, there exists n_o such that $xKn_o^c \subset U$. By theorem 4.5, x is Lagrange stable.

Theorem 4.7. *Let (X,T) be a transformation group whose phase space is first countable, phase group is hemicompact. For all points x,y of X, $y \in L(x)$ if and only if there exists a sequence (t_n) in T such that $t_n \rightarrow \infty$, $xt_n \rightarrow y$.*

Proof. ($\Rightarrow$) Since T is hemicompact, there exists a sequence (Kn) of compact subsets of T such that if K is any compact subset of T, then $K \subset Kn_o$ for some n_o. We can assume that (Kn) has the inclusion order. Let (Um) be a countable basis at y with the reverse inclusion order. Define $(n_1, m_1) \leq (n_2\ m_2)$ by $n_1 \leq n_2$, $m_1 \leq m_2$.

For each (n, m), $Um \cap xKn^c \neq \phi$ so there exists $t_{(n,m)} \in Kn^c$ such that $xt_{(n,m)} \in Um$. Take $t_n = t_{(n,n)}$. It is not hard to show that $t_n \to \infty$ and $xt_n \to y$.

($\Leftarrow$) By theorem 3.2, it is obvious.

Theorem 4.8. *Let (X,T) be a transformation group whose phase space is locally compact, first countable and phase group is hemicompact. There exists a sequence (Kn) of compact subsets of T such that if K is any compact subset of T, then $K \subset Kn_0$ for some n_o. Suppose that Kn^c is connected for all n. If $x \in X$ is Lagrange stable, then $L(x)$ is connected.*

Proof. Suppose that $L(x)$ is disconnected. $L(x) = A \cup B$ for some disjoint nonempty closed subsets A, B of X. B^c is a neighborhood of A. Since $L(x)$ is compact, A is compact. Since X is locally compact, there exists a neighborhood U of A such that $\bar{U} \subset B^c$ and $\bar{U}$ is compact. There exist $a \in A$, $b \in B$ for A,B are nonempty. By theorem 4.7, there exist sequences (t_m), (s_k) in T such that t_m, $s_k \to \infty$, $xt_m \to a$, $xs_k \to b$. Since U is a neighborhood of a and $\bar{U}^c$ is a neighborhood of b, there exist m_o, k_0 such that $xt_m \in U$, $xs_k \in \bar{U}^c$ for all $m \geq m_0$, $k \geq k_0$. For each n, there exist $m(n)$, $k(n)$ such that t_m, $s_n \in K_n{}^c$ for all $m \geq m(n)$, $k \geq k(n)$. Let $l_1 = max\ (m_0, k_0, m(1), k(1))$, $l_{i+1} = max(i+1), k(i+1), l_i+1)$. Clealy, $l_1 < l_2 < \cdots$, $tl_i, sl_i \in K_i{}^c$, $xt_{l_i} \in U$, $xs_{l_i} \in \bar{U}^c$ for all i. For each i, since $K_i{}^c$ is connected, there exists $r_{l_i} \in K_i{}^c$ such that $xr_{l_i} \in \partial U$. (xr_{l_i}) is a sequence in ∂U. Since ∂U is compact, (xr_{l_i}) has a convergent subsequence. Let $xr_{l_i} \to y \in \partial U$, $y \notin A \cup B = L(x)$. It is not hard to show that $r_{l_i} \to \infty$. Thus $y \in L(x)$, this is a contradiction. Hence $L(x)$ is connected.

Theorem 4.9. *Let (X,T) be a transformation group. If a point x of X is Lagrange stable, then any point y of $L(x)$ is Lagrange stable.*

Proof. Since $y \in L(x)$. $\overline{yT} \subset L(x)$. Thus $\overline{yT}$ is compact.

5. Poisson stability

Definition 5.1. Let (X,T) be a transformation group. A point x of X is said to be *Poisson stable* if $x \in L(x)$.

It is not hard to show that the following three statements are equivalent

$$x \in L(x),\ L(x) = \overline{xT},\ xT \cap L(x) \neq \phi$$

Theorem 5.2. *Let f: $(X,T) \to (Y,T)$ be a homomorphism of transformation groups. If $x \in X$ is Poisson stable, then $f(x)$ is Poisson stable.*

Proof. Since $x \in L(x)$, $f(x) \in f(L(x)) \subset L(f(x))$.

Theorem 5.3. *Let (X,T) be a transformation group. Suppose that X is first countable, T is hemicompact and T acts freely on X. If $x \in X$ is not Poisson stable, then $\psi_x: T \to xT$ is a homeomorphism.*

Proof. Clearly, ψ_x is a continuous bijection. Let A be a closed subset of T. $\psi_x(A) = xA$. Let y belong to the closure of xA in xT. Since X is first countable, there exists a sequence (t_m) in A such that $xt_m \to y$. Since T is hemicompact, there exists a sequence (K_n) of compact subsets of T such that if K is any compact subset of T, then $K \subset Kn_0$ for some n_0. Let us show that there exists n_0 such that for each m, there exists $m_0 \geq m$ such that $t_{m_0} \in Kn_0$. Suppose that for each n, there exists m_0 such that $t_m \in Kn^c$ for all $m \geq m_0$. It is not hard to show that $t_m \to \infty$. By theorem 3.2, $y \in L(x)$. Thus $y \in xT \cap L(x)$ and $x \in L(x)$. This is a contradiction for x is not Poisson stable. Take $m_1 = 1$

For each k, there exists $m_{k+1}\geq m_k+1$ such that $t_{m_{k+1}}\in Kn_0$. (t_{m_k}) is a sequence in Kn_0. Since Kn_0 is compact, (t_{m_k}) has a convergent subsequence. Let $t_{m_k}\to t\in Kn_0$. Clearly, $xt_{m_k}\to xt$. (t_{m_k}) is also a sequence in A. Thus $t\in\bar{A}=A$ and $yt\in xA$. Since (xt_{m_k}) is a subsequence of (xt_m), $xt_{m_k}\to y$, $y=xt\in xA$ for X is Hausdorff. Thus the closure of xA in xT contained in xA and so xA is a closed subset of xT. Hence ϕ_x^{-1} is continuous. Consequently, ϕ_x is a homeomorphism.

Let (X,T) be a transformation group. If $x\in X$ is not Poisson stable, then $xT\cap L(x)=\phi$ by definition. Since $\overline{xT}=xT\cup L(x)$, $L(x)=\overline{xT}-xT$. So $L(x)=\overline{\overline{xT}-xT}$. But if x is Poisson stable, this does not hold in general.

Theorem 5.4. *Let (X,T) be a transformation group. Suppose that X is locally compact, T is hemicompact and T acts freely on X. If $x\in X$ is Poisson stable, then $L(x)=\overline{\overline{xT}-xT}$.*

Proof. It is clear that $\overline{\overline{xT}-xT}\subset L(x)$.

Let $y\in L(x)$. Given any neighborhood U of y, we will show that $U\cap(L(x)-xT)\neq\phi$. Since X is locally compact, there exists a neighborhood V of y such that $\bar{V}\subset U$ and $\bar{V}$ is compact. Since T is hemicompact, there exists a sequence (K_n) of compact subsets of T such that if K is any compact subset of T, then $K\subset K_{n_0}$ for some n_0. It is not hard to show that $L(x)=\bigcap_{n=1}^{\infty}\overline{xK_n^c}$. $V\cap xK_1^c\neq\phi$. Thus there exists $t_1\in K_1^c$ such that $xt_1=y_1\in V$. Since T acts freely on X, $xK_1\cap xK_1^c=\phi$ so $y_1\notin xK_1$. Thus $V\cap(xK_1)^c$ is a neighborhood of y_1. There exists a neighborhood V_1 of y_1 such that $\bar{V}_1\subset V\cap(xK_1)^c$. Since $y_1=xt_1\in xT\subset\overline{xT}=L(x)$, $V_1\cap xK_2^c\neq\phi$ so there exists $t_2\in K_2^c$ such that $xt_2=y_2\in V_1$. Since $xK_2\cap xK_2^c=\phi$, $y_2\notin xK_2$. So $V_1\cap(xK_2)^c$ is a neighborhood of y_2. There exists a neighborhood V_2 of y_2 such that $\bar{V}_2\subset V_1\cap(xK_2)^c$. We proceed in this way repeatedly. Clearly $V_n\cap xK_n^c\neq\phi$ for all n, and $\bar{V}_1\supset\overline{V_1\cap xK_1^c}\supset\overline{V_2\cap xK_2^c}\supset\cdots$. Since $\bar{V}$ is compact, $\phi\neq\bigcap_{n=1}^{\infty}\overline{V_n\cap xK_n^c}$ $\subset\bigcap_{n=1}^{\infty}(\bar{V}_n\cap\overline{xK_n^c})=\bigcap_{n=1}^{\infty}\bar{V}_n\cap\bigcap_{n=1}^{\infty}\overline{xK_n^c}=\bigcap_{n=1}^{\infty}\bar{V}_n\cap L(x)$ so there exists $z\in\bigcap_{n=1}^{\infty}\bar{V}_n\cap L(x)$. Since $z\in\bar{V}_1\subset V\subset U$ and $\bar{V}_1\cap xK_1=\phi$, $z\notin xK_1$. Thus $z\notin xT$. Therefore $z\in U\cap(L(x)-xT)$. This means that $y\in\overline{L(x)-xT}$. Consequently, $L(x)=\overline{\overline{xT}-xT}$.

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