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The Exponential Smoothing and the Variance of Errors.

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1. Introduction

Recent observations should be given more weight in forecasting than older observations.

In such a situation, substitute F_t for X_{t-N}

$$F_{t+1}=(X_t+X_{t-1}+\cdots\cdots+X_{t-N+1})/N$$
$$F_t=(X_{t-1}+X_{t-2}+\cdots\cdots+X_{t-N})/N$$
$$F_{t+1}-F_t=(X_t-X_{t-N})/N$$

It becomes $\therefore\ F_{t+1}=X_t/N-X_{t-N}/N+F_t.$

$$F_{t+1}=1/N(X_t)-1/N(F_t)+F_t$$
$$=1/N(X_t)+(1-L/N)F_t$$
$$F_{t+1}=\alpha X_t+(1-\alpha)F_t \quad\cdots\cdots(1)$$

equation (1) is a general form used in computing a forecast with the method of exponential smoothing.

Where the fraction α: smoothing constant.

An alternative way of writing (1)

$$F_{t+1}=\alpha X_t+(1-\alpha)F_t=F_t+\alpha(X_t-F_t)$$

if $X_t-F_t=e_t$, then $F_{t+1}=F_t+\alpha e_t$

Equation (1) is expanded by replacing F_t with its component as follows:

$$F_{t+1}=aX_t+(1-\alpha)[\alpha X_{t-1}+(1-\alpha)F_{t+1}]$$
$$=\alpha X_t+\alpha(1-\alpha)X_{t-1}+(1-\alpha)^2F_{t+1}\cdots\cdots(2)$$

If this substitution process is repeated the result is equation (3)

$$F_{t+1}=\alpha X_t+\alpha(1-\alpha)X_{t-1}+\alpha(1-\alpha)^2X_{t-2}+\cdots+\alpha(1-\alpha)^{N-2}X_{t-N+2}+(1-\alpha)^{N-1}X_{t-N+1}\cdots\cdots(3)$$

Equation (3) can be seen that the weights applied to each of the past values decrease exponentially thus the name exponential smooth.

2. Development of Simple Exponential Smooting.

Suppose we belive that the average level of demand is not changing over time.

In this case, model process is

$$X_t=b+\varepsilon_t$$

where b: expected demand in any period.

ε_t: random component with mean 0, variance σ_ε^2

At the end of period T, we have available a demand history $X_1, X_2,\cdots X_T$, from which we wish to estimate b and σ_ε^2.

We have availed the estimate of b made at the end of the previous period $\hat{b}(T-1)$, and current period's actual demand X_T.

We want to use this information to calculate an updated estimate $\hat{b}(T)$.

The new estimate is to modify the old estimate by some fraction of the forecast error resulting from using the old estimate to forecast demand in the current period.

This forecast error is

$$e_t(T)=X_T-\hat{b}(T-1)$$

so that if α is the desired fraction.

The new estimate $\hat{b}(T)$ is

$$\hat{b}(T)=\hat{b}(T-1)+\alpha[X_T-\hat{b}(T-1)]$$

in simplify notation $\hat{b}(T)=F_t$

$$F_T=F_{T-1}+\alpha[X_T-F_{T-1}]=F_{T-1}+\alpha e_t(T)$$

where F_T: the smoothed value

Sum of weighted squared error

$$SS_E=\sum_{t=1}^{T}\beta^{T-t}(X_t-b)^2 \quad (0<\beta<1)$$

where β^{T-t}: the weight of t^{th} squared error

The estimate of b, made at the end of period T, and denoted by b(T), must satisfy

$$\left.\frac{\partial SS_E}{\partial b}\right]_{b=\hat{b}}=-2\sum_{t=1}^{T}\beta^{T-t}[X_t-\hat{b}(T)]=0$$

or

$$\hat{b}(T)\sum_{t=1}^{T}\beta^{T-t}=\sum_{t=1}^{T}\beta^{T-t}X_t$$

$$\hat{b}(T)=\frac{1-\beta}{1-\beta^T}\sum_{t=1}^{T}\beta^{T-t}X_t \cdots\cdots (4)$$

Equation (4) expresses the estimator as a function of all prior historical data.

From (4)

$$\hat{b}(T)=\frac{1-\beta}{1-\beta^T}\sum_{t=1}^{T}\beta^{T-t}X_t=\frac{1-\beta}{1-\beta^T}\left(X_T+\sum_{t=1}^{T-1}\beta^{T-t}X_t\right) \cdots\cdots ①$$

$$\hat{b}(T-1)=\frac{1-\beta}{1-\beta^{T-1}}\sum_{t=1}^{T-1}\beta^{T-1-t}X_t=\frac{1-\beta}{(1-\beta^{T-1})\beta}\sum_{t=1}^{T-1}\beta^{T-t}X_t$$

$$\sum_{t=1}^{T-1}\beta^{T-t}X_t=\frac{(1-\beta^{T-1})\beta}{1-\beta}\hat{b}(T-1)\cdots\cdots ②$$

②→①

$$\hat{b}(T)=\frac{1-\beta}{1-\beta^T}\left\{X_T+\frac{\beta(1-\beta^{T-1})}{1-\beta}\hat{b}(T-1)\right\}=\frac{(1-\beta)X_T+\beta(1-\beta^{T-1})\hat{b}(T-1)}{1-\beta^T} \cdots\cdots (5)$$

If $T\to\infty$, then $\beta^T\to 0$.

Equation (5) becomes

$$\hat{b}(T)=(1-\beta)X_T+\beta\hat{b}(T-1).$$

Letting $\alpha=1-\beta$, $\hat{b}(T)=F_t$,

$$F_T=\alpha X_T+(1-\alpha)F_{T-1}.$$

Substituting for F_{T-1} it's components

$$F_T=\alpha X_T+\alpha(1-\alpha)X_{T-1}+(1-\alpha)^2F_{T-2}$$

recursively for F_{T-k}, $k=2, 3, \cdots, T$.

$$F_T=\alpha\sum_{k=0}^{T-1}(1-\alpha)^kX_{T-k}+(1-\alpha)^TF_0$$

where F_0: the initial estimate of b.

For T sufficiently large s.t. $(1-\alpha)^TF_0\to 0$

$$E(F_T)=E\left[\alpha\sum_{k=0}^{\infty}(1-\alpha)^kX_{T-k}\right]$$

$$=\alpha\sum_{k=0}^{\infty}(1-\alpha)^kE(X_{T-k})=b\cdot\alpha\cdot\frac{1}{1-(1-\alpha)}=b.$$

Therefore, it seems reasonable to use F_T as an estimator of the unknown parameter b, that is, at time T

$$b(T)=F_T.$$

The forecast for demand in any future period $T+\tau$ would be

$$\hat{X}_{T+\tau}(T)=F_T \cdots\cdots (6)$$

In exponential smoothing, the weight given to data k periods ago is $\alpha(1-\alpha)^k$
so that
the average is

$$\alpha\sum_{k=0}^{\infty}(1-\alpha)^k\cdot k=\frac{1-\alpha}{\alpha}$$

because

$$A=\sum_{k=0}^{\infty}(1-\alpha)^k\cdot k\cdots\cdots ①$$

$$(1-\alpha)A=\sum_{k=0}^{\infty}(1-\alpha)^{k+1}k\cdots ②$$

①−②

$$\alpha A=(1-\alpha)+(1-\alpha)^2+\cdots\cdots$$

$$=\frac{1-\alpha}{1-(1-\alpha)}$$

$$A=\frac{1-\alpha}{\alpha^2}$$

$$\therefore\ \alpha\sum_{k=0}^{\infty}(1-\alpha)^kk=\alpha\cdot\frac{1-\alpha}{\alpha^2}=\frac{1-\alpha}{\alpha}$$

Thus if we wish to define an exponential smoothing system that is equivalent to an N-period moving average.

We set

$$\frac{1-\alpha}{\alpha}=\frac{N-1}{2} \quad \text{or} \quad \alpha=\frac{2}{N+1}$$

$$V(F_T)=V\left[\alpha\sum_{k=0}^{\infty}(1-\alpha)^k X_{T-k}\right]=\alpha^2\sum_{k=0}^{\infty}(1-\alpha)^{2k}V(X_{T-k})$$

$$=\alpha^2\cdot\frac{1}{1-(1-\alpha)^2}V(X_{T-k})=\frac{\alpha}{2-\alpha}\sigma_\varepsilon^2 \quad \cdots\cdots\cdots\cdots(7)$$

$$V(M_T)=V\left[\frac{1}{N}\sum_{k=0}^{N-1}X_{T-k}\right]=\frac{1}{N}\sigma_\varepsilon^2 \quad , \qquad \text{where } M_T\text{: moving average,}$$

in $V(F_T)$, for α replace $\frac{2}{N+1}$

$$V(F_T)=\left(\frac{2}{N+1}\right)\Big\backslash\left(2-\frac{2}{N+1}\right)\sigma_\varepsilon^2=\frac{1}{N}\sigma_\varepsilon^2=V(M_T)$$

The variance of exponential smoothing equal to the variance of moving average.

3. Estimation of Damand Variance.

The constant demand process is

$$X_t=b+\varepsilon_t$$

where b: expected value of demand in any period,

random component ε_t: the random deviation from the mean in period t, (somethimes "noise" component) with mean O, variance σ_ε^2

assumed $E(\varepsilon_j,\varepsilon_k)=0$ for all $j\neq k$.

The variance of forecast error, σ_e^2

For a constant demand process and simple exponential smoothing.

Then
$$\sigma_e^2=\frac{2}{2-\alpha}\sigma_\varepsilon^2$$

Proof.

$$e_{T+\tau}(T)=X_{T+\tau}-\alpha\sum_{k=0}^{\infty}\beta^k X_{T+\tau-k}$$

$X_{T+\tau}$ and X_T are iid, so, that $T+\tau\Rightarrow T$

$$e_{T+\tau}(T)=X_T-\alpha\sum_{k=0}^{\infty}\beta^k X_{T-\tau-k}$$

$$e^2=E(X_T^2)-2\alpha\sum_{k=0}^{\infty}\beta^k E(X_T X_{T-\tau-k})+\alpha^2\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\beta^{k+m}E(X_{T-\tau-k}X_{T-\tau-m})$$

Here
$$\gamma_0=\sigma_x^2=E[\{X_T-E(X_T)\}^2]$$
$$\gamma_i=E[\{X_T-E(X_T)\}\{X_{T+i}-E(X_{T+i})\}]$$

Therefore
$$E[X_T^2]=\gamma_0+\{E(X_T)\}^2=\gamma_0+b^2$$
$$E[X_T X_{T+i}]=\gamma_i+E(X_T)E(X_{T+i})$$
$$=\gamma_i+b^2$$

$$\therefore\ E(e^2)=\gamma_0+b^2-2\alpha\sum_{k=0}^{\infty}\beta^k(\gamma_{\tau+k}+b^2)+\alpha^2\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\beta^{k+m}(\nu_{m-k}+b^2)$$

$$=\gamma_0+b^2-2\alpha\sum_{k=0}^{\infty}\beta^k(\gamma_{\tau+k}+b^2)+\alpha^2\sum_{k=0}^{\infty}\beta^{2k}(\gamma_0+b^2)+\alpha^2\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\beta^{k+m}(\gamma_{m-k}+b^2),\ m\neq k,$$

if $i\neq 0$, then $\gamma_i\neq 0$,

$$=\gamma_0+b^2-2\alpha b^2\sum_{k=0}^{\infty}\beta^k+\frac{\alpha}{2-\alpha}(\gamma_0+b^2)+\alpha^2b^2\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\beta^{k+m},$$

$m\neq k,\ \ m=k+i,$

$$=\frac{2}{2-\alpha}\gamma_0+\frac{2}{2-\alpha}b^2-2b^2+2\alpha^2b^2\sum_{k=0}^{\infty}\sum_{i=1}^{\infty}\beta^{2k+i}$$

$$=\frac{2}{2-\alpha}\gamma_0+\frac{2}{2-\alpha}b^2-2b^2+2\alpha^2b^2\frac{1}{1-\beta^2}\cdot\frac{1}{1-\beta}$$

$$=\frac{2}{2-\alpha}\gamma_0+\frac{2-4+2\alpha+2-2\alpha}{2-\alpha}b^2$$

$$=\frac{2}{2-\alpha}\sigma_x^2=\frac{2}{2-\alpha}\sigma_\varepsilon^2$$

$$\therefore\ \sigma_e^2=\frac{2}{2-\alpha}\sigma_\varepsilon^2$$

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