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# Clusters and Smirnov Compactification of $E_0$-proximity Spaces

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## I. Introduction

In this paper, we deal with the compactification problem of $E_0$-*proximity spaces*. The concept of a cluster, the analogue in a proximity space of an ultrafilter, was introduced by Leader [8] and provides an alternative approach to many proximity problem. In praticular, Leader[8] obtained the compactification of a proximity space $X$ to be the family of all clusters in $X$. Since the Smirnov compactification is unique, it is evident that a one-to-one correspondence exists between clusters and ends. In this note we try to obtain the Smirnov compactification of the $E_0$-*proximity space* by using cluster.

## II. Preliminaries

The theory of proximity spaces was essentially discovered in the early 1950's by Efremovič when he axiomatically characterized the proximity relation $A$ is near $B$, which is denoted by $A\delta B$, for subsets $A$ and $B$ of any set $X$. Efremovič's axioms of proximity relation $\delta$ are as follows;

B1. $A\delta B$ implies $B\delta A$

E2. $(A\cup B)\delta C$ if and only if $A\delta C$ or $B\delta C$

E3. $A\delta B$ implies $A\neq\phi$, $B\neq\phi$.

E4. $A\not\delta B$ implies there exists a subset $E$ such that $A\not\delta E$ and $(X-E)\not\delta B$.

E5. $A\cap B\neq\phi$ implies $A\delta B$.

A binary relation $\delta$ satisfying axioms E1-E5 on the power set of $X$ is called a (Efremovič) proximity on $X$. If $\delta$ also satisfies the following;

E6. $x\delta y$ implies $x=y$

then $\delta$ is called the separated proximity relation.

**Definition 1.** Let $\delta$ be a binary relation between a set $X$ and its power set $P(X)$ such that

$E_0$1. $x\delta\{y\}$ implies $y\delta\{x\}$.

$E_0$2. $x\delta$ $(A\cup B)$ if and only if $x\delta A$ or $x\delta B$

$E_0$3. $x$ $\not\delta\phi$ for all $x\in X$.

$E_0$4. $x\in A$ implies $x\delta A$.

$E_0$5. For each subset $E\subset X$ there is a point $x\in X$ such that either $x\delta A$, $x\delta E$ or $x\delta B$, $x\delta(X\text{-}E)$, then we have $y\delta A$ and $y\delta B$ for some $y\in X$. The binary relation $\delta$ is called the $E_0$-*proximity* on $X$ iff $\delta$ saitsfies the axioms $E_0$1-$E_0$5. The pair $(X,\delta)$ is called the $E_0$-*proximity spaces.*

**Theorem 1.** *In an $E_0$-proximity spaces $(X, \delta)$ let $\delta_1$ be a binary relation on $P(X)$ defined as follows; $A\delta_1 B$ if and only if there is a point $x\in X$ such that $x\delta A$, $x\delta B$,* then $\delta_1$ *is Efremovič proximity.*

**Definition 2.** If on a set $X$ there is a topology $\zeta$ and a *$E_0$-proximity* $\delta$ such that $\zeta=\zeta(\delta)$, then $\zeta$ and $\delta$ are said to be compatible.

**Theorem 2.** *In an $E_0$-proximity space$(X, \delta)$, if $A^\delta$ is defined to be a set $\{x|x\delta A,\ x\in A\}$ for each subset $A$ of $X$, then $\delta$ is a Kuratowski closure operator. We consider the relation ultrafilter and cluster in $E_0$-proximity spaces. It is well known that a family $\mathfrak{L}$ of subets of a non-empty set $X$ is an ultrafilter if and only if the following conditions are satisfied;*

*(i) If $A$ and $B$ belong to $\mathfrak{L}$, then $A\cap B\neq\phi$.*

*(ii) If $A\cap C\neq\phi$ for every $C\in\mathfrak{L}$, then $A\in\mathfrak{L}$.*

*(iii) If $(A\cup B)\in\mathfrak{L}$, then $A\in\mathfrak{L}$ or $B\in\mathfrak{L}$.*

Now we consider the family of sets in an *$E_0$-proximity space* satisfying condition similar to (i), (ii), (iii), with nearness replacing non-empty intersection and we are led to the following definition;

**Definition 3.** A family $\sigma$ of subsets of an *$E_0$-proximity space* $(X, \delta)$ is called a cluster, iff the following conditions are satisfied;

(1) If $A$ and $B$ belong to $\sigma$, then there is a point $x\in X$ such that $x\delta A$ and $x\delta B$.

(2) If for every $C\in\sigma$, there is a point $x\in X$ such that $x\delta A$, $x\delta C$, then $A\in\sigma$.

(3) If $(A\cup B)\in\sigma$, then $A\in\sigma$ or $B\in\sigma$.

**Lemma 1.** *For each $x$ in the $E_0$-proximity space $(X, \delta)$, the family $\sigma_x=\{A\subset X|x\delta A\}$ is a cluster.*

**Lemma 2.** *If $\sigma$ is any cluster in $(X, \delta)$, then $A\in\sigma$ if and only if $\bar{A}\in\sigma$.*

**Lemma 3.** *If a cluster $\sigma$ in an $E_0$-proximity space $(X, \delta)$ is determined by an ultrafilter $\mathfrak{L}$ then $\sigma$ is a point cluster $\sigma_x$ if and only if $\mathfrak{L}$ converges to $x$. $E_0$-proximity mapping $f:(X, \delta_1)\rightarrow(Y, \delta_2)$ can be defined similar to the Efremovič proximity mapping, that is, $f$ is an $E_0$-proximity mapping iff $x\delta_1 A$ implies $f(x)\delta_2 f(A)$. It can be easily shown that $E_0$ proximity mapping is continuous.*

**Theorem 3.** *If $f$ is an $E_0$-proximity mapping from$(X, \delta_1)$ to $(Y, \delta_2)$, then to each cluster $\sigma_1$ in $X$, there corresponds a cluster $\sigma_2$ in $Y$ such that $\sigma_2=\{B\subset Y|$for each $A\in\sigma_1$, there is a point $y\in Y$ with $y\delta_2 B,\ y\delta_2 f(A)\}$.*

## III. The main theorem

Using cluster, we shall now construct the Smirnov compactification of a separated *$E_0$-proximity space*. Let $(X, \delta)$ be a separated *$E_0$-proximity space* and let $\mathfrak{X}$ denote the set of all clusters in $X$. For $A\subset X$, let $\bar{A}=\{\sigma\in\mathfrak{X}|A\in\sigma\}$. For $x\in X$, let $f(x)=\sigma_x$ (the point cluster). Then it is easy to see that (i) $f$ is a one-to-one mapping, and (ii) $f(A)\subset\bar{A}$. It is to obtain property (i) that we insist on the *$E_0$-proximity* $\delta$ being separated, for we are then assured that each point $x\in X$ is a member of one and only one cluster in $X$.

**Definition 4.** For $P\subset\mathfrak{X}$, we say that a subset $A$ of $X$ absorbs $P$ iff $A\in\sigma$ for every $\sigma\in P$, that is, $P\subset\bar{A}$, denoted $A\mapsto P$.

**Lemma 4.** The binary relation $\delta^*$ on the power set of $\mathfrak{X}$ defined by $P\delta^*\mathfrak{L}$ iff $A$ absorbs $P$ and

$B$ absorbs $\mathcal{L}$ implies that there is a point $x \in X$ such that $x\delta A$, $x\delta B$, is a separated proximity on $\mathfrak{X}$.

**proof.** E1. is clear.

E2. Suppose that $(P \cup \mathcal{L})\delta^*\mathcal{L}$ and that $P\delta^*\mathcal{L}$. Let $B \mapsto \mathcal{L}$ and $C \mapsto \mathcal{L}$, then we must show that there is some $x \in X$ with $x\delta B$, $x\delta C$. Since $P \not\delta^* \mathcal{L}$, there are sets $A$ and $D$ absorbing $P$ and $\mathcal{L}$ respectively such that for each point $x \in X$, $x \not\delta A$ or $x \not\delta D$. By definition 1-($E_0 5$) there is a subset $E \subset X$ such that for each point $x \in X$ neither $x\delta A$, $x\delta E$ nor $x\delta D$, $x\delta(X-E)$. Since $D \mapsto \mathcal{L}$ and $x \not\delta D$ or $x \not\delta (X-E)$ for each $x \in X$, $(X-E)$ belongs to no cluster in $\mathcal{L}$. Consequently, $(C-E)$ belongs to no cluster in $\mathcal{L}$. But $C=(C-E) \cup (C \cap E) \mapsto \mathcal{L}$ implies that $(C \cap E) \mapsto \mathcal{L}$. Now $(A \cup B) \mapsto (P \cup \mathcal{L})$ implies that there is some $x \in X$ with $x\delta(A \cup B)$, $x\delta(C \cap E)$. Since for each $y \in X$ $y \not\delta A$ or $y \not\delta E$, we also have for each $y \in X$ $y \not\delta A$ or $y \not\delta (C \cap E)$. Hence we have for some $x \in X$ $x\delta B$ and $x\delta(C \cap E)$, that is, $x\delta B$ and $x\delta C$. Conversely suppose that $\mathcal{L}\delta^*\mathcal{L}$, $D \mapsto (P \cup \mathcal{L})$ and $C \mapsto \mathcal{L}$. Then $D \mapsto \mathcal{L}$ and we have for some $x \in X$ $x\delta D$ and $x\delta C$. Thus $(P \cup \mathcal{L})\delta^*\mathcal{L}$.

E3. That $P\delta^*\mathcal{L}$ implies $P$ and $\mathcal{L}$ are nonempty in clear.

E4. If $P\delta^*\mathcal{L}$, there are sets $A$ and $B$ absorbing $P$ and $\mathcal{L}$ respectively such that for each $x \in X$, $x \not\delta A$ or $x \not\delta B$. By definition 1-($E_0 5$), there is a subset $E \subset X$ such that for each $x \in X$, $x \not\delta A$ *or* $x \not\delta E$ and $x \not\delta B$ and $x \not\delta (X-E)$. Since for each $x \in X$ $x \not\delta B$ or $x \not\delta (X-E)$, and $B \mapsto \mathcal{L}$, $(X-E)$ belongs to no cluster in $\mathcal{L}$. Thus $E \mapsto \mathcal{L}$. Now let $\mathcal{L}=\varepsilon$. Then $P\delta^*\mathcal{L}$ because $A \mapsto P$, $E \mapsto \mathcal{L}$ and for each $x \in X$ $x \not\delta A$ or $x \not\delta E$. Since $E$ belongs to no cluster in $(\mathfrak{X}-\mathcal{L})$. $(X-E) \mapsto (\mathfrak{X}-\mathcal{L})$. Therefore $(\mathfrak{X}-\mathcal{L})\delta^*\mathcal{L}$ since for each $x \in X$ $x \not\delta B$ or $x \not\delta (X-E)$.

E5. Suppose that $P \cap \mathcal{L} \neq \phi$ and $A \mapsto P$, $B \mapsto \mathcal{L}$. Then both $A$ and $B$ absorbs $P \cap \mathcal{L}$, so that there is some $x \in X$ with $x\delta A$ and $x\delta B$. Hence $P\delta^*\mathcal{L}$.

E6. $A \mapsto \{\sigma\}$ where $\sigma \in \mathfrak{X}$ iff $A \in \sigma$. Hence $\sigma_1 \delta^* \sigma_2$ iff $\sigma_1 = \sigma_2$. Thus $\delta^*$ is separated. Now let $\zeta^*$ be the induced toplogy on $\mathfrak{X}$ by $\delta^*$, then $\zeta^*$ is a completely reguar topology.

**Theorem** 4. *$(X, \delta)$ is homeomorphic to $f(X)$ with the subspace proximity induced by $\delta^*$, and $f(X)$ is dense in $\mathfrak{X}$.*

**proof.** We first note that $\mapsto f(B)$ iff $B \subset A^\delta$. Hence for each $\sigma \in \mathfrak{X}$, $\sigma\delta^* f(A)$ iff $C \in \sigma$ and $B \mapsto F(A)$ imply that there is some $x \in X$ with $x\delta C$ and $x\delta B$, iff $C \in \sigma$ and $A \subset B^\delta$ imply that there is some $x \in X$ with $x\delta C$ and $x\delta B$ iff $A \in \sigma$. That is, $\bar{A}$ is the $\zeta^*$ closure of $f(A)$. Since $X$ belongs to each cluster in $X$, $f(X)$ is dense in $\mathfrak{X}$. No $\sigma_x \delta^* f(A)$ iff $A \in \sigma_x$ iff $x\delta A$. Thus $X$ is homeomorphic to $f(X)$.

**Theorem** 5. $(\mathfrak{X}, \zeta^*)$ is compact.

**proof**. By lemma 3, it suffices to prove that each cluster $\zeta$ in $\mathfrak{X}$ is a point cluster. Since $f(X)$ is dense in $\mathfrak{X}$, lemma 2 implies that $f(X) \in \zeta$. From theorem 2, there is a unique cluster $\mathcal{T}'$ in $f(X)$ such that $\mathcal{T}' \subset \mathcal{T}$. But $X$ is $E_0$-*proximally* isomorphic to $f(X)$, so that there is a cluster $\sigma$ in $X$ which corresponds to $\mathcal{T}'$. Now from the proof of theorem 4, we know that $\sigma\delta^* f(A)$ iff $A \in \sigma$. Applying the theorem 3, we obtain $\{\sigma\} \in \mathcal{T} \subset \mathcal{T}$ showing that $\mathcal{T}$ is indeed a point cluster in $\mathfrak{X}$. Since $\mathfrak{X}$ is a compact Hausdorff completely regular space, we know that $P\delta^*\mathcal{L}$ iff there is some $\sigma \in \mathfrak{X}$ with $\sigma\delta^* P$ and $\sigma\delta^*\mathcal{L}$. Hence the restriction of $\delta^*$ to $\mathfrak{X} X P(\mathfrak{X})$ is the $E_0$-proximity on $\mathfrak{X}$. From this combining the above sequence of the theorem we obtain the main result of this paper.

**Theorem 6.** *Every separated $E_0$-proximity space is a dense subspace of a compact separated $E_0$-proximity space. Since $\mathfrak{X}$ has a unique compatible separated proximity, subsets $A$ and $B$ of $X$ are near iff their closures in $\mathfrak{X}$ intersect ($\mathfrak{X}$ is called the Smirnov compactification of $X$).*

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