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On the Measure Extension and Nuclear Space

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I. Introduction

One of the more important subclasses of general L^2 Kernels that have attracted considerable attention in past years is the set of nuclear or trace-class kernels, which give rise to the so-called trace-class operator (see, 1).

However, a thorough understanding of the properties of these kernels appears to be basic to the study of nuclear spaces and measures. In the first stage, some results of these efforts may refer to the reference 2.

The fundamental theorem of Minlos says that when H is a nuclear space, a finitely additive cylinder measure on H^* (the dual of H) can be extended to a σ-additive measure on the σ-algebra generated by the cylinder sets.

However it is a well-known result that one can not generally extend a cylinder measure to a σ-additive one in a Hilbert space (see, 3). Therefore, in this note we consider this problem on the σ-Hibert space (Fréchet space) in a similar way to Umemura's method (see, 3).

To process our discussions, this note falls into the following main two steps; the first step is to summarize the properties of nuclear space, together with Hopf's extension theorem, and in the second step, the relation between the extension measure and nuclear space is pointed out. In the process of developing our discussions, the detailed proofs of the lemma in section II, are omitted so that we should like to refer to the references of this note.

II. Preliminary definitions and lemmas

Let H_1 and H_2 be two Hilbrt spaces, T a completely continuous linear operator from H_1 to H_2. Let T^* denote the adjoint of T. Then T^*T is a completely continuous self-adjoint linear operator from H_1 to H_1. Moreover, for any $\varphi \epsilon H_1$, $(T^*T\varphi, \varphi) = (T\varphi, T\varphi) \geq 0$, that is, T^*T is a positive operator. According to the spectral resolution theorem for completely continuous self-adjoint operator, there is an orthonormal system of eigenvectors $\{e_n\}$ of T^*T, with corresponding eigenvalues $\lambda_n^2 > 0$, such that, for any $\varphi \epsilon H_1$,

$$T^*T\varphi = \sum_{n=1}^{\infty} \lambda_n^2 (\varphi, e_n) e_n. \tag{1}$$

Let $g_n = (1/\lambda_n) Te_n$, then

$$(g_m, g_n) = \frac{1}{\lambda_n \lambda_m} (Te_n, Te_m) = \frac{1}{\lambda_n \lambda_m} (T^*Te_n, e_m) = \delta_{nm},$$

where δ_{nm} is the kronecker delta, that is, $\{g_n\}$ is an orthonormal system in H_2. Now, for any

$\varphi \epsilon H_1$, there is a vector $\mu \perp \{e_n\}$ such that

$$\varphi = \sum (\varphi, e_n) e_n + \mu,$$

since $\mu \perp \{e_n\}$, $\mu \perp T^*T\mu$, that is, $(T\mu, T\mu) = 0$. Hence

$$T\varphi = \sum (\varphi, e_n) Te_n \text{ or, } T\varphi = \sum_{n=1}^{\infty} \lambda_n (\varphi, e_n) g_n \tag{2}$$

where $\lambda_n > 0$ and $\lim_{n\to\infty} \lambda_n = 0$.

If, in (1), we have $\sum_{n=1}^{\infty} \lambda_n^2 < \infty$, then the operator T is said to be of Hilbert-Schmidt type(written briefly as $H-S$ type); if $\sum_{n=1}^{\infty} \lambda_n < \infty$, then T is said to be a nuclear operator. Obviously, every nuclear operator is of $H-S$ type, but an operator of $H-S$ type is not necessarily nuclear. Any continuous linear operator of finite rank is, of course, nuclear. The product of two $H-S$ type operators is a nuclear operator. (see, 4).

Let H be a countable Hilbert space, with the sequence of inner product $\{(.,.)_n\}$. Let H_n be the completion of H with respect to $(.,.)_n$, and, for $m \geq n$, let T_n^m be the imbedding operator from H_m to H_n. Suppose that, for every n, there is an $m \geq n$ such that T_n^m is a nuclear operator. Then H is called a nuclear space. Since the product of two $H-S$ type operators is a nuclear operator, it is easily seen that a countable Hilbert space H is a nuclear space if and only if, for every n, there is an $m \geq n$ such that $T_n^m : H_m \to H_n$ is of $H-S$ type.

The basic properties of nuclear space are to derive in a many different ways, that is, by direct derivation from the definition or, by its stability properties (see, 5). But from now on, we apply only the following equivalent properties of them,

(H is nuclear)$\Leftrightarrow$(for every n, there is an $m \geq n$ such that the imbedding operator $T_n^m : \mathrm{H}_m \to H_n$ is of $H-S$ type)$\Leftrightarrow$(for every n, there is an $m \geq n$ such that the imbedding operator $T_m^{*n} : H_n^* \to H_m^*$ is of $H-S$ type).

Suppose that there is a sequence of subsets of a Hilbert space with the following properties;

(i) for each $n \in N$, $H_1 \supset H_2 \supset \ldots \supset H_n \supset \ldots$ and $H = {}_{n=1}\bigcap^{\infty} H_n$,

(ii) $H \subset H_n$ and H dense in H_n, (iii) $\varphi \in H$, $\|\varphi\|_1 \leq \|\varphi\|_2 \leq \cdots \leq \|\varphi\|_n \leq \cdots$.

We call H a σ-Hilbert space (Fréchet space- written briefly as F-space) if we introduced π-topology (or projective topology) induced by a countable norm system $\{\|\cdot\|_n\}$ to H. If H_n^* is the dual of H_n, then H_n is isomorphic to H_n^* by Riesz's theorem (see, 3), so H_n^* is considered as $H^* = \bigcup_n H_n^*$.

We now consider the following lemmas (see, 3, 5, 6) prior to our discussion on how a cylinder measure on $\mathcal{T}$ (that is an algebra but need not be a σ-algebra) can be extended to a σ-algebra.

Lemma 1. (**Hopf's extention theorem**) *Let a family of subsets $\mathcal{T}$ (need not be a σ-algebra) and let μ be a finitely additive measure. Then μ can be extended to the smallest of a countably additive measure defined on a σ-algebra B that contains $\mathcal{T}$ iff,*

$$\mu^*(A) = \inf \left\{ \sum_{n=1}^{\infty} \mu(A_n) \mid A \subset \bigcup_{n=1}^{\infty} A_n,\ A_n \in \mathcal{T} \right\} \tag{3}$$

The measure is regular iff for every open set u,

$$\mu(u) = \sup\{\mu(k) \mid k \subset u; k \text{ is compact}\}. \tag{4}$$

This is equivalent to; for every Borel or measurable set A,

$$\mu(A) = \inf\{\mu(A) \mid A \subset u,\ u;\ \text{open}\} \tag{4)'$$

In this case, for $\mu(A) < \infty$, we have $\mu(A) = \sup\{\mu(k) \mid k\text{:compact}, k \subset A\}$. If μ is not regular we can obtain a regularization by defining

$$\nu(A) = \inf\{\mu(u) \mid A \subset u; u\text{:open}\} \tag{5}$$

Lemma 2. *Let H be an F-space and μ a regular cylinder measure on a algebra $\mathcal{T}$ (a family of cylinder subsets) in H^*. Then μ can be extended on the smallest of σ-algebra* B *that contain $\mathcal{T}$ iff, for given any $\varepsilon > 0$, there is a neighborhood of an origin, N in H such that*

$$N^\circ \cap A = \phi,\ (A \in \mathcal{T}) \Rightarrow \mu(A) < \varepsilon, \tag{6}$$

where $\quad N^\circ = \{T \in H^* : |T(f)| \leq 1,\ {}^\forall f \in N\}$.

Now, let $T_n : H \to H_n$ be the imbedding operator (an identity operator) and ν a Gaussian measure (see, 3, 6) on H_n^*. Then we define the measure $\mathcal{T}$ on H^* as

$$\bar{\nu}(A) = v\{T_n^{*-1}(A) \mid A \in \mathcal{T},\ T_n^*: H_n^* \to H^*,\ T_n^* \text{ is the dual of } T_n\}, \tag{7}$$

where $\bar{\nu}$ is a Gaussian measure in terms of an inner product $(.,.)_n$ of H_n.

We say that cylinder measure μ is continuous iff, for any $\varepsilon > 0$ there is a neighborhood of N such that (see, 6)

$$N^\circ \cap A_f = \phi,\ A_f = \{T \mid T(f) \geq 1\} \Rightarrow \mu(A_f) < \varepsilon. \tag{8}$$

This is equivalent to; for any $\varepsilon > 0$

$$f \to 0 \text{ (in } H) \Rightarrow \lim_{f \to 0} \mu(T \mid\ |T(f)| > \varepsilon) = 0. \tag{8)'$$

Accordingly, from (7),

$$\begin{aligned}
\bar{\nu}[\{T \in H^* \mid |T(f)| \geq \alpha\}] &= \nu[\{T_n \in H_n^* \mid |(T_n, T)| \geq \alpha\}] \\
&= \frac{1}{\sqrt{2\pi}} \left\{ \int_{-\infty}^{-\alpha \setminus \|f\|_n} e^{-(1\setminus 2)x^2} dx + \int_{\alpha \setminus \|f\|_n}^{\infty} e^{-(1\setminus 2)x^2} dx \right\} \\
&= \sqrt{\frac{2}{\pi}} \int_{\alpha \setminus \|f\|_n}^{\infty} e^{-(1\setminus 2)x^2} dx \to 0 \qquad (\|f\|_n \to 0,\ n \text{ is fixed and } f \text{ is a variable})
\end{aligned}$$

Therefore we have the following lemma;

Lemma 3. *A Gaussian measure is a continuous and regular cylinder measure but it does not follow a countable additivity in general.*

Let $T: H_1 \to H_2$ be a linear operator and μ_2 a cylinder measure on $H_2^* = H_1$. Now we define the cylinder measure μ_1 as

$$\mu_1(A) = \mu_2\{T^{*-1}(A) \mid A \in \mathcal{T}\} \tag{9}$$

if μ_2 satisfy the condition (8). The result is that we following lemma from (1), lemma 1 and 2;

Lemma 4. *If T is a $H-S$ type, then μ_1 satisfy the lemma* 1.

III. Main resuit

Suppose now that $H = \bigcap_{n=1}^{\infty} H_n$ is not a nuclear space. As the definition in section Ⅱ, for any chosen n, there is an $m > n$ such that imbeding $H_n^* \to H_m^*$ is not $H-S$ type. Let μ be a Gaussian measure on H^* and μ_n a Gaussian measure on H_n^* with inner product $(.,.)_n$. Then for a cylinder set A in H^*, we have

$$\mu(A)=\mu_n(H_n^*\cap A).$$

So we may consider μ to be extended to a countable additive measure. However, set $\psi_k=T^*\varphi_k$, let P be a projective operator from H_1 into a finite dimensional subspace of H_1 generated by $\{\psi_1, \psi_2, \ldots, \psi_n\}$. Then we have $PT^*\varphi_k=T^*\varphi_k$. As TPT^* is a finite dimensional symmetric operator, there are eigenvalues λ_k $(k=1,2,\ldots,n)$ and eigenvectors $e_k\in H_2$ such that

$$\lambda_k e_k=TPT^*e_k,\quad \sum_k\|T^*\varphi_k\|_1{}^2\leq\sum_{k=1}^n\lambda_k<+\infty.$$

On the other hand, for $\psi\in H_2^*$, $\|PT^*\psi\|^2\geq r^2$ is equivalent to $\sum_k\lambda_k|(\psi,e_k)_2|^2\geq r^2$ because of an assumption that T is not $H-S$ type. Now let,

$$A=\left\{\psi\,|\sum_{k=1}^m\lambda_k-2\sqrt{\sum_{k=1}^m\lambda_k}\leq\sum_{k=1}^m\lambda_k|(\psi,e_k)_2|^2\leq\sum_{k=1}^m\lambda_k+2\sqrt{\sum_{k=1}^m\lambda_k}\right\}.$$

If $\psi\in A$, then

$$\left\{\sum_{k=1}^m\lambda_k|(\psi,e_k)_2|^2-\sum_{k=1}^m\lambda_k\right\}^2/4\sum_{k=1}^m\lambda_k\leq 1,$$

and hence, by characteristic function $\chi(\psi)=0$ $(\psi\notin A)$ or 1 $(\psi\in A)$, we have

$$1-\left\{\sum_{k=1}^m\lambda_k|(\psi,e_k)_2|^2-\sum_{k=1}^m\lambda_k\right\}^2/4\sum_{k=1}^m\lambda_k\leq\chi(\psi).$$

And let μ_2 be a Gaussian measure on H_2. Then

$$\mu_2(A)=\int_{H_2^*}\chi(\psi)\,d\mu(\psi)$$

$$\geq 1-\left(4\sum_{k=1}^m\lambda_k\right)^{-1}\int_{H_2^*}\left\{\sum_{k=1}^m\lambda_k|(\psi,e_k)_2|^2-\sum_{k=1}^m\lambda_k\right\}^2d\mu(\psi)$$

$$=1-\left(4\sum_{k=1}^m\lambda_k\right)^{-1}(2\pi)^{-m\backslash 2}\int_{R^m}\left\{\sum_{k=1}^m\lambda_k x_k^2\sum_{k=1}^m\lambda_k\right\}^2e^{-1/2(x_1^2+x_1^2\ldots+x_m^2)}dx_1\cdot dx_2\ldots dx_m$$

$$=1-\sum_{k=1}^m\lambda_k^2/2\sum_{k=1}^m\lambda_k.$$

Therefore

$$\mu_2(A)\geq 1-\sum\lambda_k^2/2\sum\lambda_k\geq\frac{1}{2}.$$

Also, if $\psi\in A$, then

$$\|T^*\psi\|_1{}^2\geq\|PT^*\psi\|_1{}^2=\sum_{k=1}^m\lambda_k|(\psi,e_k)_2|^2$$

$$\geq\sum_{k=1}^m\lambda_k-2\sqrt{\sum_{k=1}^m\lambda_k}\geq r^2.$$

Now we set $e_k=Tg_k$, $g_k\in H_1$, $\alpha=\sum_{k=1}^m\lambda_k$, and

$$A_1=\{\varphi\in H_1^*|\alpha-2\sqrt{\alpha}\leq\sum_{k=1}^{m}\lambda_k|(\varphi,g_k)_1|^2\leq\alpha+2\sqrt{\alpha}\}$$

Then for arbitrary an r-neighborhood of the origin, $N_r^{(1)}$,

$$\mu_1(A_1-N_r^{(1)})\geq\bar{\mu}_2\{T^{*-1}(A_1-N_r^{(1)})\}=\mu_2(A_2)\geq\frac{1}{2}$$

with $T^{*-1}(A_1)=A$ and A_1 being a cylinder set in H_1^*.
We arrive at the conclusion that μ_1 can not be extended to a countable additive measure. This implies immediately that μ_1 must be a Gaussian measure and T must be $H-S$ type.
Conversely, for $m>n$, imbedding: $H_n^*\to H_m^*$.

Suppose that choosing N_m as a unit sphere of H_m, then from the lemma 4 we can find the smaller ε' than a given $\varepsilon>0$ such that

$$rN_m^0\cap A=\phi \text{ implies } \mu(A)<\varepsilon'.$$

The result is that it satisfies the extension condition (Lemma 1) if it is regular and continuous. So we have the following result;

(Result) *Let H be a σ-Hilbert space and μ a Gaussian measure. Then μ satisfies the Hopf's extension theorem iff H is a nuclear space.*

Abstract

In this paper we summarize the characteristic properties of the nuclear space, and then try to establish the relation between Hopf's extension theorem and nuclear space on σ-Hilbert space.

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