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On Some Properties of $L^p(\mu)$ Space

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1. Preliminary

Throughout this note, X will denote a nonempty set, A a σ-algebra of subsets of X, and μ a positive measure on the measurable space (X, A). We derive characterizations of the measurable space (X, A, μ) such that $0<p<q$ implies either $L^p(\mu)\subset L^q(\mu)$ *or* $L^q(\mu)\subset L^p(\mu)$.

Definition 1. A set E is said to be of finite measure if $E\in A$ and $\mu(E)<\infty$.

Definition 2. A set E is said to be of σ-finite measure if E is the union of a countable collection of measuarble sets of finite measure.

Definitions 3. A μ-measurable set $E\subset X$ such that $0<\mu(E)\leq+\infty$ is called an atom whenever for any μ-measurable subset E_1 of E we have either $\mu(E_1)=\mu(E)$ or $\mu(E_1)=0$.

2. Theorems

Theorem 1. *If every set of finite measure in X is a finite pairwise disjoint union of atoms, and the measures of the atoms in X are bounded away from zero, then $L^p(\mu)\subset L^q(\mu)$ for all positive real numbers p and q such that $p<q$.*

Proof. Assume $f:X\to R$ is a function such that $a=\int_X|f|^p du<+\infty$. For every $n\in N$, we set $E_n=\{x\in X: |f(x)|>n\}$. Now

$$n^p\mu(E_n)\leq\int_{E_n}|f|^p d\mu\leq\int_X|f|^p d\mu=a<+\infty,$$

so $\mu(E_n)<+\infty$ for every $n\in N$, and $\mu(E_n)\to 0$ as $n\to\infty$. The hypothesis implies that there exists an $n_0\in N$ such that $\mu(E_n)=0$ for $n\geq n_0$. This proves that $|f(x)|\leq n_0$ for all $x\in X$ except a set of μ-measure zero. Hence,

$$\int_X|f|^q d\mu=\int_X|f|^p|f|^{q-p}d\mu\leq n_0{}^{q-p}\int_X|f|^p d\mu<+\infty.$$

This completes the proof.

Theorem 2. *If $X=E_1\cup E_2$, where $E_1\subset X$ is a measurable set such that $\mu(E_1)<+\infty$ and E_2 is either empty or an atom of finite mesure, then $L^q(\mu)\subset L^p(\mu)$ for all positive numbers p and q such that $p<q$.*

Proof. Assume $f:X\to R$ is a messurable function such that $\int_X|f|^q d\mu<+\infty$. If we set $A_n=\left\{x\in E_2: |f(x)|>\frac{1}{n}\right\}$, we havc either $\mu(A_n)=0$ or $\mu(A_n)=+\infty$. Since

$$\frac{1}{n^q}\mu(A_n)\leq\int_{A_n}|f|^q d\mu\leq\int_X|f|^q d\mu<+\infty,$$

it follows that $\mu(A_n)=0$ for every $n\in N$. Let A be the set $\bigcup_{n=1}^{\infty}A_n=\{x\in E_2: f(x)\neq 0\}$. We have

$\mu(A)=0$ and $\int_X|f|^q d\mu=\int_{E_1}|f|^q d\mu<+\infty$.

Since $\mu(E_1)<+\infty$, Hölder's inequality implies that

$$\int_X|f|^p d\mu=\int_{E_1}|f|^p d\mu\leq(\int_{E_1}(|f|^p)^{p/q}d\mu)^{p/q}(\int_{E_1}1d\mu)^{1-p/q}$$
$$=\mu(E_1)^{1-p/q}(\int_{E_1}|f|^q d\mu)^{p/q}<+\infty.$$

This completes the proof.

Theorem 3. *If $f_1, f_2, \ldots$ form a Cauchy sequence in $L^p(\mu)$ for $1\leq p<\infty$, that is, $\|f_n-f_m\|\to 0$ as $n, m\to\infty$, there is an $f\in L^p$ such that $\|f_n-f\|_p\to 0$.*

Proof. Let n_1 be an integer such that $\|f_n-f_m\|_p<1/4$ for $n, m\geq n_1$, and let $g_1=f_{n_1}$. In general, having chosen $g_1, \ldots, g_k$ and $n_1, \ldots, n_k$, let $n_{k+1}>n_k$ be such that $\|f_n-f_m\|_p<(1/4)^{k+1}$ for $n, m\geq n_{k+1}$, and let $g_{k+1}=f_{n_{k+1}}$, by lemma ((1), p. 85), g_k converges a.e. to limit function f. Given $\varepsilon>0$, choose N such that $(\|f_n-f_m\|_p)^p<\varepsilon$ for $n, m\geq N$. Fix $n\geq N$. and let $m\to\infty$ through values in the subsequence, that is, let $m=n_k$, $k\to\infty$. Then,

$$\varepsilon\geq\liminf_{k\to\infty}(\|f_n-f_{n_k}\|_p)^p=\liminf_{k\to\infty}\int_X|f-f_{n_k}|^p d\mu$$
$$\geq\int_X\liminf_{k\to\infty}|f_n-g_k|^p d\mu \qquad \text{(by Fatou's lemma)}$$
$$=(\|f_n-f\|_p)^p.$$

Thus $\|f_n-f\|_p\to 0$. Since $f=f-f_n+f_n$, we have $f\in L^p(\mu)$.

Theorem 4. *If $g_1, g_2, \ldots\in L^p(\mu)$ $(p>0)$ and $\|g_k-g_{k+1}\|_p<(1/4)^k$, $k=1, 2, \ldots$, then $\{g_k\}$ converges a.e.*

Proof. Let $A_k=\{x: |g_k(x)-g_{k+1}(x)|\geq 2^{-k}\}$. Then by Chebyshev's inequality,

$$\mu(A_k)\leq 2^{kp}(\|g_k-g_{k+1}\|_p)^p<2^{-kp}, \quad \mu(\lim_n \sup A_n)=0.$$

But if $x\notin\lim_n \sup A_n$, then $|g_k(x)-g_{k+1}(x)|<2^{-k}$ for large k, so $\{g_k(x)\}$ is a Cauchy sequence of complex numbers, and there converges.

Abstract

This note treats the inclusive relation of $L^p(\mu)$ for $0<p<q$ and the convergence of sequences in $L^p(\mu)$.

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