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Some Properties of BCK-Algebras

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1. Introduction

The notion and some properties of BCK-algebra, which is an algebraic formulation of a propositional calculus, were introduced and studied by K. Iséki (see [1], [2], [3]).

In general, it is not held in the subset A of BCK-algebra M that $x*y\in A$ imply $y*x\in A$. In this paper, we studied any non-empty subset A of BCK-algebra M, which satisfies the following properties, property:

(P_1) $0\in A$, and $x\in A$ and $x*y\in A$ imply $y\in A$.

(P_2) $0\in A$, and $(y*x)*z\in A$ and $y*z\in A$ imply $x*z\in A$.

2. Preliminaries

By a BCK-*algebra*, we mean a general algebra $M=(X;*,0)$ satisfying the following conditions:

(1) $(x*y)*(x*z)\leq z*y$,

(2) $x*(x*y)\leq y$,

(3) $x\leq x$,

(4) $x\leq y$, $y\leq x$ imply $x=y$,

(5) $0\leq x$, where $x\leq y$ means $x*y=0$.

We define $x\wedge y$ as $(y*x)*y$ for $x,y\in M$. A BCK-algebra M is *commutative* if $x\wedge y=y\wedge x$.

Throughout this paper, M will mean a BCK-algebra, and A will denote a non-empty subset of M.

Theorem 1. *In a BCK-algebra M, we have*

(1) $(x*y)*z\leq(z*x)*y$.

(2) $(x*y)*z=(x*z)*y$.

(3) $(x*y)\leq z$ *implies* $(x*z)\leq y$.

(4) $(x*y)\leq x$.

(5) $x*0=x$.

Proposition 2. *Let A satisfy* (P_1). *If* $y\in A$ *and* $y\leq x$, *then* $x\in A$.

Proposition 3. *Let* M_1, M_2 *be two BCK-algebras with* $M_1\cap M_2=\{0\}$. *We define* $x*y=y$ *if* x,y *do not belong to same algebra. Then* $M_1\cup M_2$ *is a BCK-algebra.*

In a BCK-algebra M, the subset $A(a)=\{x|x\leq a$ for a fixed $a\in M\}$ of M dces not satisfy (P_1) in general.

Theorem 4. *Any set* $A(a)$ *satisfies* (P_1), *if and only if,* $x*y\leq z$ *and* $x\leq z$ *imply* $y\leq z$ *for* $x,y,z\in M$.

Theorem 5. *If A satisfies (P_2), then (P_1) also.*

Theorem 6. *In Theorem* 4, *the subset* $\{0\}$ *of M satisfies (P_2), if and onlx if, $A(a)$ satisfies (P_1).*

3. Results

Proposition 7. *In proposition* 3, *let A, B be non-empty subsets of M_1, M_2 respectively. If A, B satisfy (P_1), then the union $A\cup B$ also.*

Proof. Let $y*x\in A\cup B$, $y\in A\cup B$. Then $y*x\in A$ or $y*x\in B$, and $y\in A$ or $y\in B$. If $y*x\in A$ and $y\in A$, or $y*x\in B$ and $y\in B$, then they are clear. Let $y*x\in A$ and $y\in B$. Suppose that $x\in M_2$. Then $y*x\in M_2$ and since $y*x\in A\subset M_1$, $y*x\in M_1\cap M_2=\{0\}$. Hence, we have $x\in B$ by Proposition 2. Suppose that $x\in M_1$. Then $y*x=x\in A$. Therefore, $x\in A\cup B$ in any case, and the proof is complete.

Proposition 8. *In Proposition* 3, *let A, B be non-empty subsets of M_1, M_2 respectively. If A, B satisfy (P_2), then the union $A\cup B$ also.*

Proof. Let $(x*y)*z\in A\cup B$, $x*z\in A\cup B$. If $x, y, z\in M_1$, then it is clear. If x, y do not belong to same algebra, then $(x*y)*z=y*z\in A\cup B$. If $x, y\in M_1$, $z\in M_2$, then $(x*y)*z=z\in M_2$, $x*z=z\in M_2$. Thus, $y*z\in B$. Therefore, $y*z\in A\cup B$ in any case, and the proof is complete.

Definition 9. Let A satisfy (P_1). A is *maximal* if it is proper and it is not a proper subset of any proper subsets satisfying (P_1).

Definition 10. In a commutative BCK-algebra M, let A satisfy (P_1). A is called *prime* if $a\wedge b\in A$ implies $a\in A$ or $b\in A$.

Remark. By Theorem 5, Definitions 9 and 10 are held in the subset A of M, which satisfies (P_2).

Lemma 11. *Let A satisfy (P_2). Then for $a\in M$, the subset $B=\{x\mid x*a\in A\}$ of M is the least set containing A and a, which satisfies (P_1).*

Proof. Let $x*y\in B$ *and* $x\in B$. Then $(x*y)*a\in A$ and $x*a\in A$ imply $y*a\in A$, and we have $y\in B$. On the other hand, $0=0*a\in A$. Hence, B satisfies (P_1). Now, let the subset C of M be a set containing A and a, which satisfies (P_1). Then for $x\in B$, since $x*a\in A\subset C$ and $a\in C$, $B\subset C$. Therefore, the subset B of M is the least set containing A and a, which satisfies (P_1).

Theorem 12. *Let A satisfy (P_2). If A is maximal, then for any elements $x, y\in M$, $x*y\in A$ or $y*x\in A$.*

Proof. Suppose that $x*y\notin A$. By Lemma 11, the subset $B=\{z\mid z*x\in A\}$ of M is a set properly containing A, which satisfies (P_1). But, since A is maximal, $B=M$. Therefore, $(x*y)*x\in A$ and since $x*x=0\in A$, we have $y*x\in A$.

Theorem 13. *Let M be a commutative BCK-algebra and A satisfy (P_1). If A is maximal, then A is prime.*

Proof. Let $x\wedge y\in A$. Then $x\wedge y=(y*x)*y=(x*y)*x\in A$ since M is commutative, and then $x*y\in A$ or $y*x\in A$ by Theorem 12. Thus, if $x*y\in A$, then $x\in A$, and if $y*x\in A$, then $y\in A$. Therefore, A is prime

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