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On Normal Weights on Von Neumann Algebras*

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1. Introduction

Let M be a von Neumann algebra. A function ϕ on the positive convex cone M^+ of M with values$\geqq 0$ finite or infinite is called a weight if ϕ satisfies the conditions:

$$(1)\ \phi(x+y)=\phi(x)+\phi(y)$$
$$(2)\ \phi(\lambda x)=\lambda\phi(x) \qquad x, y\in M^+,\ \lambda\geqq 0$$

where we use the convention: $0\cdot\infty=0$.

Suppose ϕ is a weight on M^+. Then

$$\mathcal{F}_\phi=\{x\in M^+|\phi(x)<\infty\}$$

is a face of M^+, i.e. $a, b\in\mathcal{F}_\phi,\ c\in M^+,\ c\leqq a+b \Longrightarrow c\in\mathcal{F}_\phi$, the set

$$\mathfrak{N}_\phi=\{x\in M|\phi(x^*x)<\infty\}$$

is a left ideal of M, and the set

$$\mathfrak{M}_\phi=\mathfrak{N}_\phi{}^*\mathfrak{N}_\phi=\left\{\sum_{i,j}^{n}x_i{}^*x_j\,|\,x_i,\ x_j\in\mathfrak{N}_\phi,\ n\in N\right\}$$

is the linear span of $\mathcal{F}_\phi$. Furthermore, $\mathfrak{M}_\phi$ is a hereditary subalgebra of M in the sense that any element $x\in M^+$ majorized by some element in $\mathfrak{M}_\phi{}^+$ is in $\mathfrak{M}_\phi{}^+$. We say that ϕ is semifinite if $\mathfrak{M}_\phi$ is σ-weakly dense in M. The weight ϕ is faithful if $\phi(x)=0$ implies $x=0$ for each x in M^+. ϕ is called normal if $\phi(\text{lub}x_i)=\text{lub}\phi(x_i)$ for any uniformly bounded increasing net of positive elements.

The discovery of the modular operator and the modular automorphism group associated with a normal faithful semifinite weight has led to a powerful theory—the modular theory—which is nowadays essential to the consideration of many problems concerning operator algebra. The principal result of the theory, obtained by Combes, Tomita and Takesaki, is stated as follows:

We will use the notation of Takesaki.

If ϕ is a normal faithful semifinite weight on the positive part of M, then $\mathcal{A}=\mathfrak{N}_\phi{}^*\mathfrak{N}_\phi$ turns out to be an achieved left Hilbert algebra with left von Neumann algera $\mathcal{L}(\mathcal{A})$ isomorphic to M and

$$\phi(\pi(\zeta)^*\pi(\xi))=(\xi,\zeta) \qquad \xi,\zeta\in\mathcal{A}.$$

Furthermore, there exists uniquely a one-parameter group of automorphisms $\{\sigma_t\}$ of M for which ϕ satisfies the KMS-conditions in the sense that for every x, y in $\mathfrak{M}_\phi$, there exists a bounded function $F(z)$ holomorphic in and continuous on the strip, with boundary values

$$F(t)=\phi(\sigma_t(x)y) \text{ and } F(t+i)=\phi(y\sigma_t(x)).$$

Conversely, every achieved left Hilbert algebra $\mathcal{A}$ gives rise to a normal faithful semifinite weight

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ϕ on $\mathcal{L}(\mathcal{A})^+$ by the equation

$$\phi(x)=\begin{cases}\|\xi\|^2 & \text{if } x=\pi(\xi)^*\pi(\xi) \quad \xi\in\mathcal{A}\\ +\infty & \text{otherwise.}\end{cases}$$

which makes $\mathfrak{N}_\phi{}^*\cap\mathfrak{N}_\phi$ an achieved left Hilbert algebra isomorphic to $\mathcal{A}$.

However, if ϕ is normal, then there are projections p and q in M with $p\leqq q$ such that ϕ is semifinite on qMq and faithful on $(1-p)M(1-p)$. Thus the restriction to faithful semifinite weight is for most considerations only a matter of convenience.

In [3], U. Haagerup has raised the probrem:

Let ϕ be a weight on a von Neumann algebra M, and assume that the restriction of ϕ to any commutative von Neumann subalgebra is normal. Is ϕ normal?

The purpose of the present paper is to provide conditions on M under which ϕ is normal.

2. Normality

The following lemma is due to U. Haagerup [3].

Lemma 1. *For any weight ϕ on a von Neumann algebra M, the following conditions are all equivalent:*

(1) *ϕ is completely additive, i.e. $\phi(\sum x_i)=\sum\phi(x_i)$ for any set $\{x_i\}$ of positive elements for which ϕ is defined.*

(2) *ϕ is normal.*

(3) *ϕ is σ-weakly lower semicontinuous.*

(4) *$\phi(x)=\sup_{w\in F} w(x)$, $x\in M^+$, where F is a set of positive normal functionals.*

(5) $\phi(x)=\sum\phi_i(x)$ $x\in M^+$, where $\{\phi_i\}$ is a set of positive normal functionals.

Lemma 2. ([5], Theorem 3.9) *If η is a linear mapping from a von Neumann algebra M into another such algebra N, the following conditions are equivalent:*

(1) *η is σ-weakly continuous*

(2) *for each abelian *-subalgebra $\mathcal{A}$ of M, $\eta|_{\mathcal{A}}$ is σ-weakly continuous.*

Lemma 3. ([7]) *M^+ is σ-weakly closed.*

Proof. Let M^h be the set of all hermitian elements of M and S the unit sphere of M. We first show that $M^h\cap S$ is σ-weakly closed. If $M^h\cap S$ is not closed, there exists a directed set $\{x_\alpha\}$ in $M^h\cap S$ which converges to an element $a+bi$ $(b\neq 0)$ $(a, b\in M^h)$. Suppose there exists a positive number $\lambda>0$ in the spectrum of b (otherwise, consider $\{-x_\alpha\}$). Then

$$\|x_\alpha+in1\|\leqq(1+n^2)^{1/2}<\lambda+n\leqq\|b+n\|\leqq\|a+ib+in1\|$$

for some large positive number n. Since $\{x_\alpha+in1\}$ converges to $a+ib+in1$ and belongs to $(1+n^2)S$, the compactness of $(1+n^2)^{1/2}S$ implies $a+ib+in1\in(1+n^2)^{1/2}S$. This contracts the above inequality. Hence $M^h\cap S$ is σ-weakly closed, so that by the Banach-Smulian theorem, M^h is σ-weakly closed. Since $M^+\cap S\subset M^h\cap S+1\subset M^+$, we have

$$M^+\cap S=\{M^h\cap S\}\cap\{M^h\cap S+1\}.$$

Hence $M^+\cap S$ is σ-weakly closed, so that by the Banach-Smulian theorem, M^+ is σ-weakly closed.

Theorem 4. *Let ϕ be a weight on a von Neumann algebra M such that whose restriction to any commutative von Neumann subalgebra is normal. If $\phi(1)$ is finite, then ϕ is normal.*

Proof. Let $x\in M^+$, then $x\leqq\|x\|1$. Since ϕ is order preserving, $\phi(x)\leqq\|x\|\phi(1)$. So ϕ can be extended to a linear functional $\tilde{\phi}$ on M since the positive part M^+ of M generates M. Consider a uniformly bounded increasing directed set $\{x_\alpha\}$ of positive elements in an abelian von Neumann subalgera N of M. Then

$$\tilde{\phi}(\sup_\alpha x_\alpha)=\phi|_N(\sup_\alpha x_\alpha)=\sup_\alpha(\phi|_N(x_\alpha)=\sup_\alpha(\tilde{\phi}(x_\alpha))$$

where the second equality holds because of the normality of $\phi|_N$.

Hence the restriction of $\tilde{\phi}$ to each abelian von Neumann subalgebra of M is normal. Let $x_\alpha\to x$ σ-weakly in M^+. Then by lemma 3, $x\in M^+$ and $\phi(x_\alpha)=\tilde{\phi}(x_\alpha)\to\tilde{\phi}(x)=\phi(x)$. Hence by lemma 1, ϕ is normal.

Lemma 5. ([1]) *Any abelian von Neumann algera on a separable Hilbert space is generated by a single hermitian element.*

Lemma 6. *Let M be a factor of type I_n, then we may assume that any maximal abelian von Neumann subalgebra N of M is of the form*

$$\left\{ \begin{pmatrix} a_1 & & 0 \\ & a_2 \ddots & \\ 0 & & a_n \end{pmatrix} \middle| \ a_i\in C,\ i=1,2\cdots n \right\}.$$

Proof. Since M is a factor of type I_n, M is isomorphic to $\mathcal{L}(\mathcal{H})$ where dim $\mathcal{H}=n$, whence to $M_n(C)$, the $n\times n$ matrix algebra over C. Since C^n is separable, N is singly generated by a hermitian operator T by lemma 5. By the spectral theorem, there exists an orthonormal basis of $\mathcal{H}$ consisting of eigenvectors of T. With respect to this basis, T is represented by a diagonal matrix. Since N is maximal abelian, N must be

$$\left\{ \begin{pmatrix} a_1 & & 0 \\ & a_2 \ddots & \\ 0 & & a_n \end{pmatrix} \middle| \ a_i\in C,\ i=1,2\cdots n \right\}.$$

Lemma 7. *If N is an abelian von Neumann algebra contained in $\mathcal{L}(\mathcal{H})$, then there exists a maximal abelian von Neumann algebra in $\mathcal{L}(\mathcal{H})$ containing N.*

Proof. We give a partial order to the family P of all abelian von Neumann subalgebras containing N in $\mathcal{L}(\mathcal{H})$ by set inclusion. Since the weak closure of the union of any chain is an abelian von Neumann algebra in $\mathcal{L}(\mathcal{H})$, there is a maximal element by Zorn's lemma.

Theorem 8. *If M is a factor of type I_n and if ϕ is a weight on M^+ such that the restriction of ϕ to each abelian von Neumann subalgebra is normal, then ϕ itself is normal.*

Proof. Since M is a factor of type I_n, M is isomorphic to $\mathcal{L}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Let $\alpha\in R$ be any positive number. Note that

$$\{x\in M^+|\phi(x)\leqq\alpha\}=\cup_N\{x\in N^+|\phi|_N(x)\leqq\alpha\}$$

where N runs over all abelian von Neumann subalgebras of M. By lemmm 6, 7,

$$\cup_N\{x\in N^+|\phi|_N(x)\leqq\alpha\}=\{x\in N'^+|\phi|_{N'}(x)\leqq\alpha\},\ \text{where}$$

$$N'=\left\{ \begin{pmatrix} a_1 & & 0 \\ & a_2 \ddots & \\ 0 & & a_n \end{pmatrix} \middle| \ a_i\in C,\ i=1,2,\cdots n \right\}.$$

Since $\phi|_{N'}$ is normal, it is σ-weakly lower semicontinuous by lemma 1. Hence $\{x\in M^+|\phi(x)\leqq\alpha\}$ is σ-weakly closed for any positive number α. Hence ϕ is a normal weight.

The following lemma is essentially the same as [6].

Lemma 9. *Let M be a von Neumann algera with a separating vector ξ. Then*

(a) *Let the von Neumann algebra N satisfy*

$$N\subset M,\ N\subset N', \tag{1}$$

then the orthogonal projection P on the closure of $N\xi$ is such that

$$P\xi=\xi,\ pM'p\subset(pM'p)'. \tag{2}$$

(b) *Let P be an orthogonal projection in $\mathcal{H}$ satisfying* (2), *then the von Neumann algebra $N=(M'\cup P)'$satisfies* (1).

(c) *The relation between N and P established by* (a) *and* (b) *are the inverse of each other.*

Proof. Let the von Neumann algebra N satisfy (1) and let P be the orthogonal projection on the closure of $N\xi$. We note the following facts.

(i) $P\in N'$.

[Let $x,y\in N$, we have $xpy\xi=xy\xi=pxy\xi=pxpy\xi$ and, since $N\xi$ is dense in $P(\mathcal{H})$, $xp=pxp$. By taking adjoint, $xp=px$.]

(ii) Multiplication by P yields an isomorphism

$$(M'\cup P)'\to P(M'\cup P)'.$$

[Let $x\in(M'\cup P)'$, then

$$xp=0 \Longrightarrow x\xi=0 \Longrightarrow xM'\xi=0 \Longrightarrow x=0$$

since ξ is cyclic for M'.]

(iii) $P(M'\cup P)'=P(PM'P)'$.

[$P(M'\cup P)'=(M'\cup P)''=\{PxP|_{P(H)}\,|x\in M\cup P\}'=\{PxP|_{P(H)}\,|x\in M'\}'=P(PM'P)'$.]

(iv) $PN=P(PN)'=P(PM'P)''=P(PM'P)'$.

[The restriction of PN to $P(\mathcal{H})$ is abelian and has the cyclic vector ξ, thus it is maximal abelian, hence $PN=P(PN)'$. The set $PM'P$ restricted to $P(\mathcal{H})$ commutes with PN and has the cyclic vector ξ, therefore

$$P(PN)'\supset P(PM'P)'' \text{ or } P(PM'P)'\supset PN$$

and since it is maximal, $P(PM'P)''=P(PM'P)'$.]

(v) $N=(M'\cup P)'$.

[(1) yields $N\subset(M'\cup P)'$. Thus by (ii) and (iv), we have

$$P(M'\cup P)'\subset PN=P(M'\cup P)'=P(PM'P)'.$$

This gives

$$PN=P(M'\cup P)'.$$

By (ii),

$$N=(M'\cup P)'.$$

Part (a) of the theorem and one half of part (c) follows from (iv) and (v) respectively.]

Let now P be an orthogonal projection in $\mathcal{H}$ satisfying (2). We note the following facts.

(vi) $P(PM'P)''=P(PM'P)'$.

[The restriction of $(PM'P)''$ to $P(\mathcal{H})$ is abelian and has the cyclic vector ξ. Hence it is maximal abelian.]

(vii) $P(PM'P)'=P(M'\cup P)'$.

[The proof is the same as for (iii).]

(viii) Multiplication by P yields an isomorphism

$$(M'\cup P)'\to P(M'\cup P)'.$$

[The proof is the same as for (ii).]

(ix) The closure of $(M'\cup P)'\xi$ is the range of P.

[Because $(M'\cup P)'\xi=P(M'\cup P)'\xi=P(PM'P)''\supset PM'\xi$ by (vi), (vii).]

It follows from (vi), (vii) that $(M'\cup P)'$ is abelian, proving part (b) of the theorem. The second half of (c) follows from (iv). Now we prove the main theorem.

Theorem 10. *Let M be a von Neumann algebra on $\mathcal{H}$ with a separating vector ξ, and the number of orthogonal projections in $\mathcal{L}(\mathcal{H})$ satisfying*

$$P\xi=\xi,\ \ PM'P\subset(PM'P)'$$

be finite, then a weight ϕ is normal if the restriction $\phi|_N$ of ϕ to each abelian von Neumann subalgebra $N\subset M$ is normal.

Proof. Let α be any positive number. Then

$$\{x\in M^+|\phi(x)\leqq\alpha\}=\bigcup_N\{x\in N^+|\phi|_N(x)\leqq\alpha\}$$

where N runs over all abelian von Neumann subalgebras of M.

By lemma 9, the right side of the equality is a finite union. The σ-weak closedness of each $\{x\in N^+|\phi|_N(x)\leqq\alpha\}$ completes the proof.

Remark. Theorem 8 is a special case of Theorem 10.

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