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A Property of Restricted Lie Algebra

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1. Introduction

A Lie algebra FL is called a free Lie algebra on $X=\{x_1,\dots,x_n\}$ if, given any mapping ϕ of X into a Lie algebra M, there exists a unique homomorphism $\psi: FL\to M$ extending ϕ.

This algebra FL is constructed by the Lie algebra $F(L)$ generated by X.

Here $F(L)$ is the free algebra generated by X with the bracket operation $[a,b]=ab-ba$ for each $a,b\in F(L)$.

The Friedrichs' theorem is a useful criterion for Lie elements in study of the free Lie algebra over a field of characteristic 0.

This paper will introduce an analogue of criterion in case of characteristic $p\neq 0$.

2. Preliminaries

Lemma 1 (Friedrichs). *Let $F=k(x_1,\dots,x_n)$ be the free algebra generated by the x_i over a field of characteristic* 0.

Let δ be the diagonal mapping of F, i.e., the homomorphism of F into $F\otimes F$ such that $x_i\delta=x_i\otimes 1+1\otimes x_i$.

Then $a\in F$ is a Lie elements, i.e., $a\in FL$ if and only if $a\delta=a\otimes 1+1\otimes a$.

Definition. A *restricted Lie algebra L* of characteristic $p\neq 0$ is a Lie algebra with the operation $a\to a^{(p)}$ $(a\in L)$ satisfying the following three condition.

R1) $\forall\alpha\in k,\ \forall a\in L,\ (\alpha a)^{(p)}=\alpha^p a^{(p)}$.

R2) $(a+b)^{(p)}=a^{(p)}+b^{(p)}+\sum_{i=1}^{p-1}S_i(a,b)$

where $iS_i(a,b)$ is the coefficient of λ^{i-1} in $a(\mathrm{ad}(\lambda a+b))^{p-1}$ (λ an indeterminant).

R3) $[a,b^{(p)}]=a(\mathrm{ad}b)^p$.

Lemma 2. *Let FL be the free Lie algebra generated by a set $X=\{x_1,\dots,x_n\}$. If we define $a^{(p)}=a^p$ for every $a\in FL$, then FL is a resticted Lie algebra.*

proof. R1), R2) are easily verified with some rigorous but elementary calculation.

$$\text{In } FL,\ [a,b^{(p)}]=ab^p-b^pa \text{ and}$$

$$a(\mathrm{ad}b)^p=ab^p-b^pa+\sum_{i=1}^{p+1}(-1)^i\binom{p}{i}b^iab^{p-i},$$

The last term of the right in second equation is 0.

Thus R3) is also hold,

3. Main Theorem

Theorem. *Let $F=k(x_1,\dots,x_n)$ be the free algebra over a field k of characteristic $p\neq 0$. Let δ be as in Lemma 1.*

Then $a\in F$ is in the restricted Lie algebra generated by the x_i if and only if $a\delta=a\otimes 1+1\otimes a$.

Proof. $[a\otimes 1+1\otimes a,\ b\otimes 1+1\otimes b]=[ab]\otimes 1+1\otimes[ab]$ implies that the set elements a satisfying $a\delta=a\otimes 1+1\otimes a$ is a subalgebra of $F(L)$.

This includes the x_i, hence it contains FL.

Let $y_1, y_2,\dots$ be a basis for FL. Since F is the universal enveloping algebra of FL, the elements $y_1^{k_1}y_2^{k_2}\dots y_m^{k_m}$, $k_i\geqq 0$ form a basis for F. Hence the products

$$(y_1^{k_1}y_2^{k_2}\cdots y_m^{k_m})\otimes(y_1^{l_1}y_2^{l_2}\cdots y_n^{l_n})$$

form a basis for $F\otimes F$.

$$\begin{aligned}(y_1^{k_1}y_2^{k_2}\cdots y_m^{k_m})\delta&=(y_1\otimes 1+1\otimes y_1)^{k_1}(y_2\otimes 1+1\otimes y_2)^{k_2}\dots(y_m\otimes 1+1\otimes y_m)^{k_m}\\&=y_1^{k_1}y_2^{k_2}\dots y_m^{k_m}\otimes 1+k_1y_1^{k_1-1}y_2^{k_2}\dots y_m^{k_m}\otimes y_1\\&\quad+k_2y_1^{k_1}y_2^{k_2-1}\dots y_m^{k_m}\otimes y_2+\dots+k_my_1^{k_1}\dots y_m^{k_m-1}\otimes y_m+(*)\end{aligned}$$

where (*) is a linear combination of base elements of the form

$$y_1^{j_1}y_2^{j_2}\dots y_s^{j_s}0y_1^{l_1}y_2^{l_2}\dots y_t^{l_t}\ \text{with}\textstyle\sum l_i>1.$$

In order that a shall be a linear combination of the base elements of the form $y_1^{k_1}\dots y_m^{k_m}\otimes 1$ and $1\otimes y_1^{j_1}\dots y_s^{j_s}$, it is necessary that in the expression for a in terms of the only base elements $y_1^{k_1}\dots y_m^{k_m}$ with one $k_i=1$ and all the other $k_j=0$ or np.

This means that a is a linear combination of the y_i and y_j^{np}.

But $y_j^{np}=(y_j^p)^n$ FL. Hence $a\delta=a\otimes 1+1\otimes a$ if and only if $a\in FL$ as a restricted Lie algebra.

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