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Quotient Reflective Hulls in *Top*

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1. Introduction

In the study of topological structure, the monosource is the most essential for the basic construction. It is well known that the category theory gives the convenient language and tool for the study of almost all mathematical structure.

In this paper, we try to use those concepts for the study of some specific spaces. We consider a full subcategory $\underline{A}$ of *Top*, define *mon*($\underline{A}$) and we study their internal characterizations and categorical properties for various categories $\underline{A}$. For the terminologies, refer to [Herrich & Strecker, *Category Theory*].

2. Definitions and Theorems

Proposition 1. *Top is (quotient, monosource) category.*

Proof. Let $(f_i : X \to X_i)_{i\in I}$ be a source of continuous maps, let $R = \bigcap_{i\in I} \ker(f_i)$, and $Y = X/R$ be the quotient space. Let $q : X \to Y$ be the quotient map. Then since $\ker(q) \subseteq \ker(f_i)$, there is a map $m_i : Y \to X_i$ such that $f = m_i \circ q$ for each $i \in I$. Since q is final, each m_i is continuous and hence $(m_i : Y \to X_i)_{i\in I}$ is a monosource.

Thus, $X \xrightarrow{f_i} X_i = X \xrightarrow{q} Y \xrightarrow{m_i} X_i$ $(i \in I)$ is a (quotient,monosource) factorization. For a quotient morphism e, and a monosource $(m_i)_{i\in I}$, let

$$\begin{array}{ccc} X & \xrightarrow{\quad e \quad} & Z \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g_i} \\ Y & \xrightarrow[\quad m_i \quad]{} & Y_i \end{array}$$

be a commutative diagram.

Let's show that $\ker(e) \subset \ker(f)$.

Let $(x, x') \in \ker(e)$, that is, $e(x) = e(x')$. Then $g_i(e(x)) = g_i(e(x'))$ for all $i \in I$. Since $m_i \circ f(x) = m_i \circ f(x')$ for all $i \in I$ and $(m_i)_{i\in I}$ is a monosource, $f(x) = f(x')$, that is, $\ker(e) \subset \ker(f)$. Consequently, there exists a unique map $h : Z \to Y$ with $f = h \circ e$. Since $m_i \circ h \circ e = m_i \circ f = g_i \circ e$ and e is epi, $m_i \circ h = g_i$.

Finally, since $m_i \circ h = g_i$, g_i is continuous, and $(m_i)_{i\in I}$ is initial, it follows that h is continuous.

Theorem 1. *Let $\underline{A}$ be a full, isomorphism-closed subcategory of Top. Then $\underline{A}$ is quotient reflective, that is, each reflection is given by a quotient map if and only if $\underline{A}$ is closed under the formation of* *monosources.*

Proof. Let $(f_i : X \to X_i)_{i \in I}$ be a monosource with $X_i \in \underline{A}$ for all $i \in I$ and e be the $\underline{A}$-reflection of X. Then there exists a unique continuous map. $h_i : A \to X_i$ with $h_i \circ e = f_i$ for each $i \in I$. Since *Top* is (quotient, monosource)-category,there is a unique continuous map k such that $k \circ e = 1_X$ and $f_i \circ k = h_i$ for all $i \in I$. Hence $e \circ k \circ e = e \circ 1_X = 1_A \circ e$. Since e is a reflection of X, $e \circ k = 1_A$. This implies that e is a homomorphism and hence $X \in \underline{A}$.

Conversely, for any $X \in Top$, let $(f_i : X \to X_i)_{i \in I}$ be the source of all continuous maps with $X_i \in \underline{A}$ and let $X \xrightarrow{f_i} X_i = X \xrightarrow{e} A \xrightarrow{m_i} X_i$ be the (quotient, monosource)-factorization of $(f_i)_{i \in I}$, that is, for each $i \in I$ $f_i = m_i \circ e$ and e is quotient and $(m_i)_{i \in I}$ is a monosource. Since $\underline{A}$ is closed under the formation of monosource, $A \in \underline{A}$.

Let's show that $e : X \to A$ is in fact the $\underline{A}$-reflection of X. For any $Y \in \underline{A}$ and a continuous map $f : X \to Y$, there is an $i \in I$ such that $X \xrightarrow{f} Y = X \xrightarrow{f_i} Y$. By the above factorization of (f_i), there is a continuous map $m_i : A \to Y$ with $f = f_i = m_i \circ e$. Since e is epimorphism, such a continuous map m_i is unique.

Theorem 2. *Let $\underline{A}$ be a full, isomorphism-closed subcategory of Top. Then $\underline{A}$ is quotient reflective if and only if $\underline{A}$ is productive, hereditary, and for $(X, \tau) \in \underline{A}$ and $\tau \subset \tau'$, $(X, \tau') \in \underline{A}$.*

Proof. Since $\underline{A}$ is quotient reflective, $\underline{A}$ is closed under the formation of monosources. Since for $A_i \in \underline{A} (i \in I)$ and a subspace X of $A \in \underline{A}$, $(p_{r_i} : \Pi A_i \to A_i)_{i \in I}$ and $e : X \to A$ are monosource, $\prod_{i \in I} X_i \in \underline{A}$ and $X \in \underline{A}$ and since $1_X : (X, \tau') \to (X, \tau)$ is monomorphism, $(X, \tau') \in \underline{A}$.

Conversely, let $(f_i : X \to X_i)_{i \in I}$ be a monosource with $X_i \in A$ and $X = (X, \tau)$. We may assume that I is a set. Let τ' be initial topology on X with respect to $(f_i)_{i \in I}$. Then $\tau' \subseteq \tau$. Since $(\prod_{i \in I} f_i)(X)$ is a subspace of $\prod_{i \in I} X_i$, $(\prod_{i \in I} f_i)(X) \in \underline{A}$, and since (X, τ') is homemorphic with $(\prod_{i \in I} f_i)(X)$, $(X, \tau') \in \underline{A}$. On the other hand,since $1_X : (X, \tau) \to (X, \tau')$ is monomorphism, $(X, \tau) \in \underline{A}$ and hence, $\underline{A}$ is quotient reflective.

Notation 1. Let $\underline{A}$ be full subcategory of *Top* and let $mon(\underline{A})$ be $\{X \in Top \mid$ there is a monosource $\{f_i : X \to A_i\}_{i \in I}$ in *Top* such that for all $i \in I$, $A_i \in \underline{A}\}$.

Note that in *Top*, $\{f_i : X \to A_i\}_{i \in I}$ is a monosource if and only if $\{f_i : X \to A_i\}_{i \in I}$ seperates points of X, that is, if $x \neq y$ in X, then there is $i_0 \in I$ with $f_{i_0}(x) \neq f_{i_0}(y)$, equivalently, if $f_i(x) = f_i(y)$ for all $i \in I$, then $x = y$.

Theorem 3. *Let $\underline{A}$ be a full, isomorphism-closed subcategory of Top. Then $mon(\underline{A})$ is the smallest quotient reflective subcategory of Top containing $\underline{A}$.*

Proof. Since the composition of monosource is again a monosource, $mon(\underline{A})$ is closed under formation of monosources and hence $mon(\underline{A})$ is quotient reflective in *Top*.
It is obvious that $\underline{A} \subset mon(\underline{A})$. Let $\underline{B}$ be a quotient reflective subcategory of *Top* containing $\underline{A}$. Since $\underline{B}$ is closed under the formation of monosources and $\underline{A} \subset \underline{B}$, $mon(\underline{A}) \subset \underline{B}$. That is, $mon(\underline{A})$ is the smallest quotient reflective subcategory of *Top* containing $\underline{A}$.

Corollary 1. *Let $\underline{A}$ be a subcategory of Top. Then $mon(\underline{A})$ is the extremal epireflective hull of $\underline{A}$*

in Top.

Corollary 2. 1) *$mon(\underline{A})$ is closed under the formation of initial monosources.*
2) *$mon(\underline{A})$ is productive and hereditary.*
3) *For any $(X,\tau)\in mon(\underline{A})$ and $\tau'\supseteq\tau$, (X,τ') also belongs to $mon(\underline{A})$.*

Theorem 4. *For any $\underline{A}\subseteq Top$ and $X\in Top$, the followings are equivalent.*
1) *$X\in mon(\underline{A})$.*
2) *There is a family $(A_i)_{i\in I}$ in $\underline{A}$ and there is a map $f: X\to \Pi A_i$ which is 1-1 continuous.*
3) *There is a 1-1 continuous map $f: X\to Y$ such that $Y\in Epi\ R(\underline{A})$, where $Epi\ R(\underline{A})$ is the epireflective hull of $\underline{A}$ in Top.*

Proof. 1)$\Longrightarrow$2) Since $X\in mon(\underline{A})$, there is a monosource $(f_i: X\to A_i)_{i\in I}$ such that for all $i\in I$, $A_i\in\underline{A}$. By the definition of product ΠA_i, there is a 1-1 continuous map f such that $p_{r_i}\circ f=f_i$ for all $i\in I$.
2)$\Longrightarrow$3) Let $Y=\Pi A_i$. Then $Y\in Epi\ R(\underline{A})$ and hence $f: X\to Y$ is a 1-1 continuous map.
3)$\Longrightarrow$1) Since $Epi\ R(\underline{A})\subset mon(\underline{A})$, 1-1 continuous map $f: X\to Y$ is a monosource, hence $X\in mon(\underline{A})$.

Notation 2. 1) Let *CHaus* be the category of completely Hausdorff spaces and continuous maps.
2) Let *Comp Sep* be the category of completely seperated spaces and continuous maps.
3) Let O_v be the category of normal spaces and continuous maps.

Corollary 3. 1) *$X\in CHaus$ if and only if there is a 1-1 continuous map $f: X\to Y\in CReg$.*
2) *$X\in Comp\ Sep$ if and only if there is a 1-1 continuous map $f: X\to Y$ with $Y\in Zdim$.*

Proof. Since $mon(\{I\})=CHaus$, 1) is obvious. Since $mon(\{0,1\})=Comp\ Sep$, 2) is obvious.

Remark. $mon(O_v)\supsetneqq O_v$. Suppose $mon(O_v)=O_v$. Since $mon(O_v)$ is productive, O_v is productive, which is a contradiction.

Example 1. *Let $X\in\underline{A}$, where X contains an indiscrete subspace A with $|A|\geqslant 2$. Then $mon(\underline{A})=Top$.*

Proof. Let $x_0, y_0\in A$ with $x\neq y$. Let $Y\in Top$ and for any distinct pair $a,b\in Y$, let $f_{a,b}: Y\to X$ be a continuous map defined by $f_{a,b}(a)=x_0$ and $f_{a,b}(x_0)=y_0$ for $x\neq a$. Then $(f_{a,b}: Y\to X)$ is a monosource. Hence $Y\in mon(\underline{A})$.

Example 2. $mon(Haus)=Haus$.

Proof. Pick $X\in Haus$, then $l_X: X\to X$ is a monosource. Hence X belongs to $mon(Haus)$, so that $Haus\subset mon(Haus)$. Take any monosource $(f_i: X\to X_i)_{i\in I}$ with $X_i\in Haus$ $(i\in I)$. Let $x\neq y$ in X. Then there is $i_0\in I$ such that $f_{i_0}(x)\neq f_{i_0}(y)$ in X_{i_0}. Since X_{i_0} is Hausdorff, there is a disjoint nbds U_{i_0}, V_{i_0} of $f_{i_0}(x)$, $f_{i_0}(y)$. Then $f^{-1}(U_{i_0})$ and $f^{-1}(V_{i_0})$ are disjoint nbds of x and y respectively, so that X is a Hausdorff space.

Example 3. *$mon(\{R\})=mon(\{I\})=CHaus$, where R is the real line with the usual topology, I is the unit interval $[0,1]$ with the usual topology, and CHaus is the category of completely Hausdorff spaces.*

Proof. Let $\{f_i: X\to R\}_{i\in I}$ be a monosource. Since R is completely regular, X belongs to $mon(\{\underline{I}\})$. Conversely, let $\{g_i: X\to I\}_{i\in I}$ be a monosource in *Top*. Then for $x\neq y$ in X, there is $i_0\in I$ such that

$g_{i_0}(x) \neq g_{i_0}(y)$ in $[0,1]$. Let $j:[0,1]\to R$ be an inclusion map. Then $j\circ g_{i_0}(x) \neq j\circ g_{i_0}(y)$ in R. Hence X belongs to $mon(\{\underline{R}\})$. Let $X \in mon(\{\underline{I}\})$ *i.e.*, there is a monosource $\{f_i : X\to[0,1]\}_{i\in I}$. Hence for $x\neq y$ in X, there is $i_0 \in I$ such that $f_{i_0}(x) \neq f_{i_0}(y)$ in $[0,1]$. Since $[0,1]$ is completely regular, there is a continuous function $u:[0,1]\to[0,1]$ such that $u(f_{i_0}(x))=0$, $u(f_{i_0}(y))=1$. Then X is a completely Hausdorff space. Then for any distinct points x, y, there is a continuous $v: X\to[0,1]$ such that $v(x)=0$, $v(y)=1$. Thus X belongs to $mon([0,1])$. The two point space $\{0,1\}$ with only non-trivial open set $\{1\}$ is called the Sierpinski space and denoted by S.

Example 4. *Let S be a Sierpinski space. Then mon $(\{S\})=Top$.*

Proof. Let $\{f_i : X\to S\}_{i\in I}$ be a monosource and let $x\neq y$ in X. Then there is $i_0\in I$ such that $f_{i_0}(x)=0$, $f_{i_0}(y)=1$ (or $f_{i_0}(x)=1$, $f_{i_0}(y)=0$) so that $f_{i_0}^{-1}(1)$ is an open nbd of y which does not contain x_0. Hence X is a T_0-space. Conversely, let X be a T_0-space, and let $x\neq y$ in X. Assume there is a nbd U of x which does not contain y. Then χ_U is a continuous map from X to S and $\chi_U(x)\neq\chi_U(y)$. Hence $C(X,S)$ is a monosource. Thus X belongs to $mon(\{S\})$.

Let D be the two point discrete space $\{0,1\}$, *Comp Sep* be the category of completely seperated spaces and *Tdisc* be the category of totally disconnected spaces.

Example 5. $mon(\{\underline{D}\})=Comp\ Sep \subseteq disc$.

Proof. Let $\{f_i : X\to D\}_{i\in I}$ be a monosource. Thus for every distinct points x, y in X, such that $f_i(x)=0$ and $f_i(y)=1$. Let $f_i^{-1}(0)=U_i$ and $f_i^{-1}(1)=V_i$. Then $U_i\cap V_i=\phi$. Fix x in X and let $y\neq x$ in X. Then $\bigcap_{y\in X-\{x\}} U_i=\{x\}$, where U_i is clopen. Hence X belongs to *Comp Sep*. Let X be a completely seperated space, then $C(X,D)$ is a monosource and hence X belongs to $mon(\{\underline{D}\})$.

Finally, we must show that $Comp\ Sep\subseteq Tdisc$. Let X be a completely seperated space and suppose $C_X \supset \{x,y\}$. Since $\{x\}=\bigcap\{U \mid U$ is a clopen nbd of $x\}$, there is a clopen nbd U of x such that $y\notin U$, and hence $\{x,y\}$ is disconnected which is a contradiction. Thus $C_X=\{x\}$. Thus X is a totally disconnected space.

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