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Integral Representation of C^∞ Solutions of Linear Partial Differential Equations with the Cannonical Form

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1. Introduction

In this paper Ω will denote an open neighborhood $\{(x,t)\,|\,|x|<r,\ |t|<\delta\}$ of the origin of R^2.

We consider the general linear P.D.E with the cannonical form

$$L=\partial/\partial t+ib(x,t)\partial/\partial x$$

where $b(x,t)$ is a real valued C^∞ function in Ω.

The linear P.D.E L is said to satisfy the condition (P) in if for any $x\in(-r,r)$, the function $t\mapsto b(x,t)$ does not change sign and satisfy condition (P_1) if

(i) $b(x,t)>0$ for any $(x,t)\in\Omega$ with $x\neq 0$

(ii) $b(0,t)=0$ for any t, $|t|<\delta$ and satisfy (P_2)

if $b(x,t)\neq 0$ for any $(x,t)\in\Omega$.

We now assume that L satisfies (P_1) or (P_2). It implies that L satisfies (P). That $Lu=f$ is locally solvable follows immediately from the general criteria for the local solvability of a linear P.D.E due to Nirenberg-Treves [2].

We also assume L satisfies the followings;

$$Lz=\partial z/\partial t+ib(x,t)\partial z/\partial x=0,$$
$$Re\ z_x>0$$

has a C^∞ solution in Ω.

We shall represent a C^∞ solution in an integral form in a neighborhood of the origin. When $b(x,t)$ is real analytic, the same result is established in Treves[3].

2. Integral Representation

Let $z=z(x,t)$ be a C^∞ solution of

$$Lz=\partial z/\partial t+ib(x,t)\partial z/\partial x=0,$$
$$Re\ z_x>0 \qquad (1)$$

in Ω. This is the generalization of $x-it^2/2$ of the Mizohata operator.

We write $z(x,t)=\xi(x,t)+i\eta(x,t)$

where ξ and η are real valued. Thus $\partial\xi/\partial x\neq 0$ in Ω. Therefore we have the right to change variables

$$y=\xi(x,t),\quad s=t \qquad (2)$$

in Ω.

Let $z=y+i\phi(y,s)$

where $\phi(y,s)=\eta(x,t)$ real valued, C^∞ and $\partial y/\partial x\neq 0$ in Ω.

In (y, s) coordinates we have

$$L=\partial/\partial t+ib(x,t)\partial/\partial x$$
$$=\partial/\partial s+\{\partial y/\partial t+ib(x,t)\partial y/\partial x\}\ \partial/\partial y$$
$$=\partial/\partial s+\lambda(y,s)\ \partial/\partial y.$$

But $Lz=0$, that is,

$$0=L(y+i\phi)=i\phi_s+\lambda(1+i\phi_y).$$

From this,

$$\lambda=\partial y/\partial t+ib\partial y/\partial x=-i\phi_S/(1+i\phi_y).$$

Since y is a real valued function

$$b(x,t)=-(1+\phi_y^2)^{-1}\ (\partial y/\partial x)^{-1}\phi_S(y,s). \tag{3}$$

Let the C^∞ map $\psi:(x,t)\to(y,s)$ be the local coordinate change as defined by (2) and r be the positive numbers such that

$$\{(y,s)\mid |y-k|<\bar{r},\ |s|<\bar{\delta}\}\subset\psi(\Omega)$$

where k is a constant number as follows:

We first assume that L satisfies condition (P_1). We claim ψ maps $\{(0,t)\mid |t|<\delta\}$ into $\{(k,s)\mid |s|<\bar{\delta}\}$ in (y,s) plane. In fact, from the condition (P_1) $b(0,t)=0$ for any t, $|t|<\delta$. Therefore $\partial/\partial t(\xi+i\eta)=0$ for any t, $|t|<\delta$. So $\xi(0,t)=k$ (a constant as above) for any t, $|t|<\delta$.

From (3), $\phi_s(k,s)=0$ for any s, $|s|<\bar{\delta}$.

So $\phi(k,s)=\alpha$ (a constant).

Since ψ is a bijective map, the inverse image of $\{(y,s)\mid |y-k|<\bar{r},\ |s|<\bar{\delta}\}$ for any fixed $y\neq k$ under ψ is entirely contained in $\{(x,t)\in\Omega\mid x>0\}$ or $\{(x,t)\in\Omega\mid x<0\}$.

Note that for any fixed $y\neq k$, $|y-k|<\bar{r}$, then the map $s\mapsto\phi(y,s)$ is a strictly increasing function in the interval $|s|<\bar{\delta}$.

Note that (i) of condition (P_1) is a special case of four other kinds of signs in $\{(x,t)\in\Omega\mid x>0\}\cup\{(x,t)\in\Omega\mid x<0\}$.

For instance, if $b(x,t)<0$ for any $(x,t)\in\Omega$, $x\neq0$, then the map $s\mapsto\phi(y,s)$ is a strictly decreasing function for any fixed $y\neq k$, $|y-k|<\bar{r}$.

Now we subdivide the open rectangles

$$|y-k|<\bar{r},\quad |s|<\bar{\delta} \tag{4}$$

as a union of $I=\{(k,s)\mid |s|<\bar{\delta}\}$

and a open rectangles $R^+=\{(y,s)\mid k<y<k+\bar{r},\ |s|<\bar{\delta}\}$

and $R^-=\{(y,s)\mid k-\bar{r}<y<k,\ |s|<\bar{\delta}\}$.

We note that the ranges of the map $z=y+i\phi(y,s)$ restricted to the rectangle (4) as follows:

(i) z maps I to the single point $k+i\alpha$

(ii) z maps the rectangles R^+ and R^- homeomorphically onto open sets θ_1 and θ_2 of the complex plane C which are entirely contained, respectively, in the strip $k<Re\ z<k+\bar{r}$ and in the strip $k-\bar{r}<Re\ z<k$.

We shall denote by A the image of the rectangle (4) under ψ.

Let now $f(x,t)$ be any C^∞ function in R^2 with support contained in

$$V=\psi^{-1}\{(y,s)\mid |y-k|<\bar{r},\ |s|<\bar{\delta}\}.$$

We note the equation

$$Lu=\partial u/\partial t+ib(x,t)\partial u/\partial x=f \text{ is equivalent to}$$
$$(\partial/\partial s+\lambda(y,s)\partial/\partial y)(u(y,s)-\int_{-\bar{\delta}}^{S} f(y,\sigma)d\sigma)=$$
$$-\lambda(y,s)\int_{-\bar{\delta}}^{S}(\partial f/\partial y)(y,\sigma)d\sigma. \tag{5}$$

Here $f(y,s)=f(x(y,s),t)$ etc.

For the simplicity we shall set

$$v(y,s)=u(y,s)-\int_{-\bar{\delta}}^{s} f(y,\sigma)d\sigma,$$
$$g(y,s)=-\lambda(y,s)\int_{-\bar{\delta}}^{s}(\partial f/\partial y)(y,\sigma)d\sigma.$$

$\lambda=-i\phi_s/1+i\phi_y$ vanishes identically on the vertical line segment I (where $\phi_s=0$).

Now we transform v and g to the set A under the map $z=y+i\phi(y,s)$.

Since $g=0$ on I and z is a homeomorphism on R^+ and R^-, the transfered function $g(z)$ can be extended by 0 outside of A and is equal to compactly supported function of L^1 class, with a compact support contained in $\bar{A}$.

The equation (5) becomes

$$(\partial\bar{z}/\partial s+\lambda(y,s)\partial\bar{z}/\partial y)(\partial\tilde{v}/\partial\bar{z})-\tilde{g}$$

where v denotes $v(y,s)$ as a function of z.

But since $\lambda(y,s)(\partial z/\partial y)=-\partial z/\partial s$, we have $\partial\bar{z}/\partial s=-\bar{\lambda}(\partial\bar{z}/\partial y)$.

Therefore (5) reads to

$$2i\ (Im\lambda)^{\sim}(\partial\bar{z}/\partial y)(\partial\tilde{v}/\partial\bar{z})=\tilde{g}. \tag{6}$$

Moreover, since $\partial\bar{z}/\partial y=1-i\phi_y$ and $Im\lambda=-\phi_s/1+i\phi_y{}^2$ (6) equivalent to

$$[(-2i/1+i\phi_y)\phi_s]^{\sim}\partial\tilde{v}/\partial\bar{z}=\tilde{g} \tag{7}$$

or

$$\partial\tilde{v}/\partial\bar{z}=i/2[(1+i\phi_y)g/\phi_s]^{\sim}=\left[-1/2\int_{-\bar{\delta}}^{s} f(y,\sigma)d\sigma\right]^{\sim}. \tag{8}$$

(8) is a inhomogeneous Cauchy-Riemann equation whose solution is given by

$$\tilde{v}=1/2\pi i\iint F(\varphi)/z-\varphi\ d\bar{\varphi}\wedge d\varphi$$

where

$$F(z)=i/2[(1+i\phi_y)g/\phi_s]^{\sim}$$
$$=\left[-1/2\int_{-\bar{\delta}}^{S} f(y,\sigma)d\sigma\right]^{\sim}. \tag{9}$$

To revert (9) to (y,s) coordinates, we set

$$\varphi=y'+i\phi(y',s').$$

Then we have

$$d\bar{\varphi}\wedge d\varphi=2i\phi_{s'}dy'\wedge ds'$$

and hence

$$v(y,s)=1/2\pi\iint_{R^2}\phi_{s'}(y',s')k(y',s')/[y-y'+i(\phi(y,s)-\phi(y',s'))]dy'\wedge ds' \tag{10}$$

where

$$k(y,s)=\int_{-\bar{\delta}}^{s}(\partial f/\partial y)(y,\sigma)d\sigma.$$

Since $v(y,s)$ is the pullback via $(y,s)\mapsto y+i\phi(y,s)$ of $\tilde{v}$ which is locally L^1 function, $v(y,s)$ is well defined and in fact, a C^∞ solution.

Then for any $f\in C_0^\infty(V)$ a C^∞ solution of $Lu=f$ in V is given by the pullback via the map $\phi:(x,t)\longrightarrow(y,s)$ defined in (2) of a C^∞ solution

$$u(y,s)=-1/2\pi\iint_{R_2}\phi_{s'}(y',s')k(y',s')/[y-y+i(\phi(y,s)-\phi(y',s'))]dy'\wedge ds'$$
$$+\int_{-\delta}^{s}f(y,\sigma)d\sigma$$

where

$$k(y,s)=\int_{-\delta}^{s}(\partial f/\partial y)(y,\sigma)d\sigma.$$

So far we considered only the case when L satiesfies (P_1). When L satisfies (P_2), the argument is much simpler, as z is a homeomorphism on the entire rectangle (4) in this case.

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