韓國數學敎育學會誌<數學敎育>
1984. 12, Vol. XXIII, No. 1

Journal of the Korea Society of Mathematical Education
Dec. 1984, Vol. XXIII, No. 1

On the Exact Triangle of Direct Sums of Torsion Products

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0. Introduction

Let $_RM$ be the category of R-modules and R-homomorphisms, where R is a commutative ring with unity. Also, let COMP be the category of all chain complexes and chain transformations. Then it is already known that there exists the exact triangle of homology modules of differential modules induced by chain complexes C, D and E (Proposition 4).

The main object of this paper is to construct the exact triangle of direct sums of torsion products (Theorem 8). Most of notations in this paper are taken from [2].

1. Preliminaries

Definition 1. Let X be an R-module and (C, ∂) a chain complex:

$$C: \cdots\cdots \longrightarrow C_{n+1} \xrightarrow{\partial} C_n \xrightarrow{\partial} C_{n-1} \longrightarrow \cdots\cdots$$

Then C is called a *projective resolution* of X if

(1) $C_{-1}=X$

(2) $C_n=0$ for every $n<-1$

(3) C_n is a projective R-module for every $n\geq 0$.

Let X and Y be arbitrarily given R-modules. Select any projective resolution C of the module X

$$C: \cdots\cdots \longrightarrow C_{n+1} \xrightarrow{\partial} C_n \xrightarrow{\partial} C_{n-1} \longrightarrow \cdots\cdots$$

and consider its tensor product $C\otimes Y$ which is the sequence

$$C\otimes Y: \cdots\cdots \longrightarrow C_{n+1}\otimes Y \xrightarrow{\partial_*} C_n\otimes Y \xrightarrow{\partial_*} C_{n-1}\otimes Y \longrightarrow \cdots\cdots$$

where ∂_* stands for the tensor product $\partial\otimes i$ of the homomorphism ∂ and the identity endomorphism i of the module Y. Since $\partial_*\circ\partial_*=(\partial\otimes i)\circ(\partial\otimes i)=(\partial\circ\partial)\otimes(i\circ i)=0\otimes i=0$, $C\otimes Y$ is a semi-exact sequence and so $C\otimes Y$ is a chain complex. Thus for every integer n, the n-dimensional homology module $H_n(C\otimes Y)$ of $C\otimes Y$ is defined.

Proposition 2. *For any two projective resolution C, D of the module X,*

$$C: \cdots\cdots \longrightarrow C_{n+1} \xrightarrow{\partial} C_n \xrightarrow{\partial} C_{n-1} \longrightarrow \cdots\cdots$$

$$D: \cdots\cdots \longrightarrow D_{n+1} \xrightarrow{\delta} D_n \xrightarrow{\delta} D_{n-1} \longrightarrow \cdots\cdots$$

we have $H_n(C\otimes Y)\approx H_n(D\otimes Y)$ for every integer n.

Proof. The proof may be found in [2. p. 131].

The above module $H_n(C\otimes Y)$ depends essentially only on the integer n and given R-modules X, Y. Thus we make the following definition.

Definition 3. For every integer n, the R-module $H_n(C\otimes Y)$ is said to be the *n-dimensional torsion product* of the given modules X, Y and is denoted by the symbol $Tor_n^R(X, Y)$.

Proposition 4. *Let*

$$O\to C\to D\to E\to O$$

be a given short exact sequence in COMP. Then there exists an exact triangle of homology modules of differential modules induced by chain complexes C, D and E:

$$\begin{array}{ccc} H(X) & \longrightarrow & H(Y) \\ & \nwarrow \quad \swarrow & \\ & H(Z) & \end{array}$$

,where $X=\bigoplus_{n\in Z} C_n$, $Y=\bigoplus_{n\in Z} D_n$ *and* $Z=\bigoplus_{n\in Z} E_n$.

Proof. It follows from [3].

Proposition 5. *If* $O\longrightarrow A'\overset{h}{\longrightarrow}A\overset{p}{\longrightarrow}A''\longrightarrow O$ *is exact in* ${}_RM$, *then for any R-module K, the sequence*

$$A'\otimes K\longrightarrow A\otimes K\longrightarrow A''\otimes K\longrightarrow O$$

is exact in ${}_RM$.

Proof. It follows from [1. p. 35].

Definition 6. Let A, B and K be R-modules. Then K is said to be a *flat module* if for every monomorphism $h: A\to B$,

$$h\otimes i_k: A\otimes K\to B\otimes K$$

is monic, where i_k is the endomorphism of K.

Proposition 7. (Horseshoe Lemma) *Consider the diagram in* ${}_RM$

$$\begin{array}{ccccccccc} & & \downarrow & & & & \downarrow & & \\ & & P_1' & & & & P_1'' & & \\ & & d_1'\downarrow & & & & \downarrow d_1'' & & \\ & & P_0' & & & & P_0'' & & \\ & & \varepsilon'\downarrow & & & & \downarrow \varepsilon'' & & \\ 0 & \longrightarrow & A' & \overset{h}{\longrightarrow} & A & \overset{p}{\longrightarrow} & A'' & \longrightarrow & 0 \\ & & \downarrow & & & & \downarrow & & \\ & & 0 & & & & 0 & & \end{array}$$

where the columns are projective resolutions and the row is exact. Then there exists a projective resolution of A and chain transformations so that the columns form an exact sequence in COMP.

Proof. It follows from [1. p. 187].

2. Main theorem

Theorem 8. *Let*

$$O\longrightarrow A'\overset{h}{\longrightarrow}A\overset{p}{\longrightarrow}A''\longrightarrow O$$

be a given short exact sequence in ${}_RM$ *and K a flat module. Then there exists an exact triangle of direct sums of torsion products, that is,*

$$\begin{array}{ccc} \bigoplus_{n\in Z} Tor_n^R(A',K) & \longrightarrow & \bigoplus_{n\in Z} Tor_n^R(A,K) \\ \nwarrow & & \swarrow \\ & \bigoplus_{n\in Z} Tor_n^R(A'',K) & \end{array}$$

Proof. By Proposition 7, there exists an exact sequence

$$O \longrightarrow C \longrightarrow D \longrightarrow E \longrightarrow O \text{ in } COMP,$$

where C, D and E are projective resolutions A', A and A'' respectively. This means that, for every integer $n \in \boldsymbol{Z}$,

$$O \longrightarrow C_n \longrightarrow D_n \longrightarrow E_n \longrightarrow O$$

is exact in ${}_RM$.

By Proposition 5 and Definition 6,

$$O \longrightarrow C_n \otimes K \longrightarrow D_n \otimes K \longrightarrow E_n \otimes K \longrightarrow O$$

is exact in ${}_RM$. Thus

$$O \longrightarrow C \otimes K \longrightarrow D \otimes K \longrightarrow E \otimes K \longrightarrow O$$

is exact in *COMP*. By Proposition 4, there exists an exact triangle of homology modules of differential modules induced by chain complexes $C\otimes K$, $D\otimes K$ and $E\otimes K$:

$$\begin{array}{ccc} H(\bigoplus_{n\in Z}(C_n\otimes K)) & \longrightarrow & H(\bigoplus_{n\in Z}(D_n\otimes K)) \\ \nwarrow & & \swarrow \\ & (H\bigoplus_{n\in Z}(E_n\otimes K)) & \end{array}$$

Also we obtain $H(\bigoplus_{n\in Z}(C_n\otimes K)) = \bigoplus_{n\in Z} H_n(C\otimes K) = \bigoplus_{n\in Z} Tor_n^R(A', K)$. Similarly $H(\bigoplus_{n\in Z}(D_n\otimes K)) = \bigoplus_{n\in Z} Tor_n^R(A, K)$ and $H(\bigoplus_{n\in Z}(E_n\otimes K)) = \bigoplus_{n\in Z} Tor_n^R(A'', K)$. This proof completes.

References

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