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On π-Injective and p-Injective Modules

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1. Introduction

Throughout this paper, we assume that rings are associative and have identity elements. Also, every module is left and unitary.

Let M be a left R-module. A left R-module Q is called M-injective if every homomorphism of any R-submodule of M into Q can be extended to a homomorphism of M into Q. An R-module M is said to be p-injective if, for any principal ideal P of R, any left R-homomorphism $g: P \to M$, there exists $y \in M$ such that $g(b) = by$ for all $b \in P$. An R-module M is said to be π-injective if for every pair M_1, M_2 of (non-zero) submodules of M with $M_1 \cap M_2 = (0)$, each projection $\pi_i : M_1 \oplus M_2 \to M$, $i=1,2$; can be lifted to an endomorphism of M. Obviously, any injective, quasi-injective module is π-injective. But there exists a module which is π-injective and not quasi-injective[3].

In this paper, we consider rings whose p-injective left modules are π-injective.

2. Main Theorems

First, we begin with the following lemma[3, p. 148].

Lemma 2.1. *Let M and N be R-modules such that $M \oplus N$ is π-injective. Then M is N-injective and N is M-injective.*

Theorem 2.2. *The following conditions are equivalent:*

(1) *R is left Noetherian ring whose p-injective left modules are injective.*

(2) *Every p-injective left R-module is injective.*

(3) *Every p-injective left R-module is quasi-injective.*

(4) *Every p-injective left R-module is π-injective.*

proof. It is obvious that (1) implies (2), (2) implies (3) and (3) implies (4). Assume (4). Let M be a p-injective left R-module, $E(R)$ the injective hull of R. Since the direct sum of p-injective modules is p-injective, $E(R) \oplus M$ is p-injective. Now, by assumption, $E(R) \oplus M$ is p-injective. Then, by Lemma 2.1, M is $E(R)$-injective, and hence injective. Note that any direct sum of p-injective left R-modules is p-injective. Thus any direct sum of injective left R-modules is p-injective, and hence injective, by above proof. Hence R is left Noetherian [2, Theorem 20.1]. And this completes the proof.

Corollary 1. *The following conditions are equivalent:*

(1) *R is a principal left ideal ring.*

(2) *Every p-injective left R-module is π-injective and every finitely generated left ideal of R is principal.*

proof. Let R be a principal left ideal ring. Then Baer's criterion implies that p-injective R-module is injective and also π-injective. Conversely, assume (2). Then, by Theorem 2.2, R is Noetherian. hence every ideal is finitely generated. Thus (2) implies (1).

Corollary 2. *If the sum of any two p-injective left R-modules is π-injective, then R is left Noetherian, left hereditary.*

proof. Since the assumption implies (4) in Theorem 2.2, R is left Noetherian and every p-injective left module is injective. Recall that R is left hereditary iff the sum of any two injective left R-modules is injective. Thus the proof is complete.

Theorem 2.3. *The following conditions are equivalent for a commutative ring R:*

(1) *R is Noetherian and hereditary.*

(2) *The sum of any two p-injective R-modules is injective.*

(3) *The sum of any two p-injective R-modules is p-injective and π-injective.*

proof. It is obvious that (2) implies (3). By Corollary 2 to Theorem 2.2, (3) implies (1). And (1) and (2) are equivalent [5, Proposition 5].

Theorem 2.4. *The following conditions are equivalent:*

(1) *Any π-injective left R-module is injective.*

(2) *The direct sum of any two π-injective left R-modules is π-injective.*

(3) *Any direct sum of π-injective left R-modules is π-injective.*

(4) *R is Noetherian and each π-injective left R-module is injective.*

proof. It is obvious that (1) implies (2), (3) implies (2) and (4) implies (1). By Proposition 2.6 in [1], (2) and (4) are equivalent. Finally, recall that if R is Noetherian, then any direct sum of injective left R-modules is injective. Now (3) follows from (4) easily.

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