

A Multiproduct Facility-in-Series Production Planning Model

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Abstract

A deterministic multiproduct, facility-in series multiperiod production planning model is analyzed, where each period demand for the product of a facility appear in a fixed proportion of that for the product of the immediately following facility. The model considers concave production and inventory costs, which can depend upon the production in different facilities. No backlogging is allowed. It is shown that the model is represented via a single source network, which facilitates development of efficient dynamic programming algorithms for computing the optimal production schedule.

1. INTRODUCTION

A multifacility, multiproduct production and inventory problem has been analyzed in Zangwill [2], where the individual facilities are linked together to form an acyclic network. In the network, each facility can receive inputs from either raw materials or lower numbered facilities, but cannot receive inputs from itself or higher numbered facilities. Similarly, each facility can supply only higher numbered facilities or market requirements for its own product, so that the product in each facility can be different. The first facility receives raw materials only and the last facility supplies market requirements only.

Assuming that the joint production costs among facilities and each inventory cost are piecewise concave, Zangwill [2] characterized the dominant set composed of production schedules satisfying exact requirements. It was shown that a production schedule contained in the set is optimal, which minimizes the total piecewise concave cost function. Even if the concept of piecewise concavity proved to be quite general, he developed efficient algorithms for only two special models; one for a parallel system, each facility of which supplies only market requirements and no other facilities, and the other one for a series system, where each facility supplies no market requirements and no facilities other than its immediately following one, so that only the last facility supplies market requirements. In the latter case, Zangwill [3] has applied the concept of concave cost network analysis to an efficient dynamic programming algorithm development for finding the optimal production schedule in all echelons.

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In this paper, a multiproduct, multifacility production model is considered that is essentially for a series system of two facilities. The first facility can receive inputs from raw material only, but produce two different items: one for the last facility and the other one for its own market requirements, where in each period the respective production volumes are in the ratio of β to α . The last facility can receive inputs from the first facility and supply the market requirements only. As an example, an oil refinery system can be taken into account, which produces two different items, say gasoline and the remainder. Gasoline may be required for automobiles, but the remainder may be required for the further processing to produce some fine chemicals. It is assumed that the two different products have the market requirements of α to β ratio in each period. Furthermore, a FIFO issuing policy is assumed.

2. THE MODEL FORMULATION

Let r_{ij} , $r_{ij} \geq 0$, be the market requirements for facility j 's product in period i , ($j=1, 2$; $i=1, 2, \dots, N$), where N is the number of periods under consideration. For this study, r_{i1} and r_{i2} will be called the first stage demand and the second stage demand in period i , respectively. It is assumed that all requirements r_{ij} are fixed and known in advance. Furthermore, r_{i1} and r_{i2} are required in the fixed ratio of α to β in each period i . Let x_{ij} , $x_{ij} \geq 0$, be the production completed in period i facility j and I_{ij} the inventory at the end of period i in facility j .

Assume that no backlogging is permitted so that $I_{ij} \geq 0$ for all i and j . Let $I_{i1} = I_{i1}(1) + I_{i1}(2)$, where $I_{i1}(1)$ and $I_{i1}(2)$ represent the inventory amount of stocks in period i at facility 1 for the first stage demands and the second stage demands, respectively. Then, the equations relating the production and inventory are

$$I_{i-1,1}(1) - I_{i1}(1) + x_{i1} * \left(\frac{\alpha}{\alpha + \beta} \right) = r_{i1}, \quad \dots(1)$$

$$I_{i-1,1}(2) - I_{i1}(2) + x_{i1} * \left(\frac{\alpha}{\alpha + \beta} \right) = x_{i2}, \quad \dots(2)$$

$$I_{i-1,2} - I_{i2} + x_{i2} = r_{i2} \quad \dots(3)$$

$i=1, 2, \dots, N,$

where $I_{01} = I_{02} = I_{N1} = I_{N2} \equiv 0$

Eqs (1) and (2) are summarized in Ed(4),

$$I_{i-1,1} - I_{i1} + x_{i1} - x_{i2} = r_{i1} \quad (4)$$

These equations assume that production is for all practical purposes instantaneous.

Let $P_{ij}(x_{ij})$ be the cost of producing x_{ij} units, and let $H_{ij}(I_{ij})$ be the cost of holding I_{ij} units in stock. All costs are assumed concave on the closed interval $(0, \infty)$ and $P_{ij}(0) = H_{ij}(0) = 0$. Let $x_j = (x_{1j}, x_{2j}, \dots, x_{Nj})$ be the production schedule for facility j . The vector $x = (x_1, x_2) = (x_{11}, x_{21}, \dots, x_{N1}, x_{12}, x_{22}, \dots, x_{N2})$ is the schedule for the entire system.

Given certain fixed nonnegative market requirements for each of the two facilities over the next N periods, the problem is to find a production schedule X , called optimal, which minimizes the concave cost function

$$F(X) = \sum_{i=1}^N \{P_{i1}(x_{i1}) + P_{i2}(x_{i2}) + H_{i1}(I_{i1}(1)) + H_{i1}(I_{i1}(2)) + H_{i2}(I_{i2})\} \quad \dots(5)$$

subject to

$$\sum_{i=1}^N x_{i1} = \sum_{i=1}^N (r_{i1} + r_{i2}),$$

$$I_{i-1,1} - I_{i1} + x_{i1} - x_{i2} = r_{i1},$$

$$I_{i-1,2} - I_{i2} + x_{i2} = r_{i2}, (i=1, 2, \dots, N)$$

$$I_{01} = I_{02} = I_{N1} = I_{N2} = 0,$$

$$I_{i1}, I_{i2}, x_{i1}, x_{i2} \geq 0, \forall i,$$

where I_{ij} is actually a linear function of the production vector X .

The constraints of Eq(5) can be represented as a single source network which is depicted in Fig. 1.

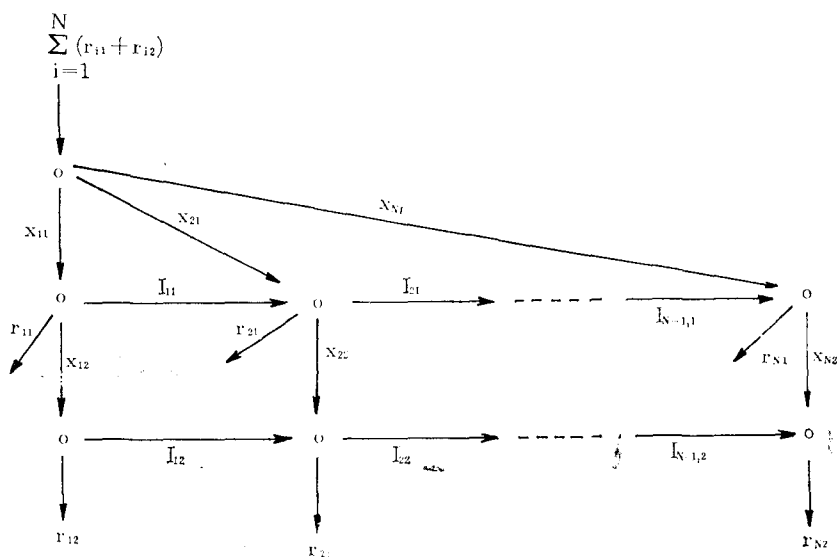


Fig. 1. The Single Source Network

3. DETERMINING AN OPTIMAL PRODUCTION SCHEDULE

Since I_{ij} are linear in X , the feasible sets are bounded and polyhedral. They are hence compact and convex with a finite number of extreme points. In fact, with the multi-echelon system represented as a network, determining the optimal production schedule for the production system is equivalent to finding the corresponding network optimal flow. As some optimal flow is an extreme flow, the search for an optimal flow will begin by characterizing the extreme flows. Denote by $I_{i1}(1)$ and $I_{i1}(2)$ the inventories at the end of period i ($i=1, 2, \dots, N$) from the production of facility 1 for the first and the second stage requirements, respectively.

Theorem 1.

Consider an extreme flow in the single source network. Assume that $x_{i1}(1) = \alpha/\beta x_{i1}(2)$ for

all i , where $x_{i1}(1)$ and $x_{i1}(2)$ represent the goods of facility 1 for the first and the second stage requirements, respectively. Then, it holds that $I_{i1}(1)=0$ iff $I_{i1}(2)=0$ and $I_{i2}=0$. Furthermore, $I_{i1}(1)>0$ iff either $I_{i1}(2)>0$ or $I_{i2}>0$.

Proof. It is known that for every period l , $r_{l1} = \left(\frac{\alpha}{\beta}\right)r_{l2}$. If $I_{i1}(1)=0$, it holds, under the given assumption, that

$$\sum_{l=1}^i x_{l1} * \left(\frac{\alpha}{\alpha+\beta}\right) = \sum_{l=1}^i r_{l1} \quad \dots(6)$$

Then, the total inventory in the system for the second stage demand is computed as follows;

$$\begin{aligned} I_{i1}(2) + I_{i2} &= \sum_{l=1}^i x_{l1} * \left(\frac{\beta}{\alpha+\beta}\right) - \sum_{l=1}^i r_{l2} \\ &= \sum_{l=1}^i x_{l1} - \sum_{l=1}^i \left[r_{l1} + r_{l1} * \left(\frac{\beta}{\alpha}\right) \right], \text{ from Eq. (6),} \\ &= 0. \end{aligned}$$

This implies that $I_{i1}(2)=0$ and $I_{i2}=0$, since $I_{i2}(2) \geq 0$ and $I_{i2} \geq 0$.

On the other hand, if $I_{i1}(2)=0$ and $I_{i2}=0$, it leads then to $I_{i1}(2) + I_{i2}=0$, so that

$$\sum_{l=1}^i x_{l1} * \left(\frac{\beta}{\alpha+\beta}\right) = \sum_{l=1}^i r_{l2}. \text{ Thence,} \quad \dots(7)$$

$$\begin{aligned} I_{i1}(1) &= \sum_{l=1}^i x_{l1} * \left(\frac{\alpha}{\alpha+\beta}\right) - \sum_{l=1}^i r_{l1} \\ &= \sum_{l=1}^i x_{l1} - \sum_{l=1}^i \left[x_{l1} * \left(\frac{\beta}{\alpha+\beta}\right) + r_{l2} * \left(\frac{\beta}{\alpha}\right) \right], \text{ from Eq. (7),} \\ &= 0. \end{aligned}$$

This completes the proof for the first statement.

Now, we will show that the second statement holds. If $I_{i1}(1)>0$, then

$$\sum_{l=1}^i x_{l1} * \left(\frac{\alpha}{\alpha+\beta}\right) > \sum_{l=1}^i r_{l1}. \quad \dots(8)$$

Therefore,

$$\begin{aligned} I_{i1}(2) + I_{i2} &= \left\{ \sum_{l=1}^i x_{l1} * \left(\frac{\beta}{\alpha+\beta}\right) - \sum_{l=1}^i r_{l2} \right\} \\ &> \left\{ \sum_{l=1}^i x_{l1} - \sum_{l=1}^i \left[x_{l1} * \left(\frac{\alpha}{\alpha+\beta}\right) + x_{l1} * \left(\frac{\alpha}{\alpha+\beta}\right) * \left(\frac{\beta}{\alpha}\right) \right] \right\}, \text{ from Eq. (8),} \\ &= 0. \end{aligned}$$

This means that either $I_{i1}(2)>0$ or $I_{i2}>0$, since $I_{i1}(2) \geq 0$ and $I_{i2} \geq 0$, and in addition both $H_{i1}(I_{i1}(2))$ and $H_{i2}(I_{i2})$ are concave.

Finally, if either $I_{i1}(2)>0$ or $I_{i2}>0$, then it leads to $I_{i1}(2) + I_{i2}>0$, so that

$$\sum_{l=1}^i x_{l1} * \left(\frac{\beta}{\alpha+\beta}\right) > \sum_{l=1}^i r_{l2}. \quad \dots(9)$$

Therefore,

$$I_{i1}(1) = \left\{ \sum_{l=1}^i x_{l1} * \left(\frac{\alpha}{\alpha+\beta}\right) - \sum_{l=1}^i r_{l1} \right\}$$

$$\begin{aligned}
&> \left[\sum_{i=1}^i x_{i1} - \sum_{i=1}^i \left[x_{i1} * \left(\frac{\beta}{\alpha + \beta} \right) + x_{i1} * \left(\frac{\beta}{\alpha + \beta} \right) * \left(\frac{\alpha}{\beta} \right) \right] \right], \text{ from Eq. (9),} \\
&= 0.
\end{aligned}$$

Thus, the proof is completed.

Since the network in Fig.1 has a single source, an extreme flow will have the form that there can be at most one positive input to any node. This fact and the results of Theorem 1 will now be exploited in Theorem 2 to determine the form of an extreme flow. Theorem 2 can be easily proved by contradiction. Points in time where the inventory level is zero will be called "regeneration points." It is then noted that for every time period where production is scheduled, the start of the period (end of the prior period) is a regeneration point.

Theorem 2.

Let the end of period m and the end of period n be regeneration points in an extreme flow for facility 1; that is, $I_{m1}(1)=0$ and $I_{n1}(1)=0$. Then,

(a) For facility 1, the extreme flow is

$$x_{m+1,1} = \sum_{i=m+1}^n r_i, \text{ where } r_i = r_{i1} + r_{i2}, \text{ and so}$$

$$I_{i1}(1) = \sum_{i=t+1}^n r_{i1}, \quad m+1 \leq t \leq n-1.$$

(b) For facility 2, either one of the following production schedules is an extreme flow:

(i) $x_{m+1,2} = \sum_{i=m+1}^n r_{i2}$, and so

$$I_{i2} = \sum_{i=t+1}^n r_{i2}, \quad m+1 \leq t \leq n-1.$$

(ii) $x_{m+1,2} = r_{m+1,2}, \dots, x_{m+\lambda-1,2} = r_{m+\lambda-1,2}, x_{m+\lambda,2} = \sum_{i=m+\lambda}^n r_{i2}$, for a λ in the range of from 2 to

$$n-m, \text{ and so, } I_{i1}(2) = \sum_{i=t+1}^n r_{i2} \text{ for } m+1 \leq t \leq m+\lambda-1 \text{ and } I_{i2} = \sum_{i=t+1}^n r_{i2} \text{ for } m+\lambda \leq t \leq n-1.$$

Under certain conditions, the given problem is equivalent to a single-product, single-facility problem treated in Wagner and Whitin [1]. Some of such conditions will be specified in Theorem 3.

Theorem 3.

The production scheduling problem is equivalent to a singleproduct, single-facility problem with the demands $r_i = r_{i1} + r_{i2}$, if either one of the following holds:

- (a) For each production set-up ($x > 0$) at facility 2, the set-up cost function, $S_i(x)$, satisfies $S_i(x) \geq H_{i2}(r_{i2})$, $\forall i$.
- (b) $H_{i1}(I_{i1}(2)) \geq H_{i2}(I_{i2})$, $\forall i$.

Proof.

From the second statement of Theorem 1, it follows that whenever facility 1 is due production for the first stage demands, facility 2 should be setup for production for the second stage

demands. Furthermore, from the last statement of Theorem 1, when $I_{i1}(1) > 0$, it is possible to have either $I_{i1}(2) > 0$ or $I_{i2}(2) > 0$. If $I_{i1}(2) > 0$ is decided, then it will be required from Theorem 1 that $x_{i2} = r_{i2}$ or else r_{i2} will not be satisfied. However, by the assumption, $S_i(r_{i2}) \geq H_{i2}(r_{i2})$, so that the production of $x_{i2} = r_{i2}$ can not lead to an optimal solution. Therefore, it is necessary to have $I_{i2} > 0$ rather than $I_{i1}(2) > 0$ in period i .

Likewise, the second statement is sufficient to the conclusion. This completes the proof.

4. AN ALGORITHM

Along with the properties of the extreme flow discussed above, regeneration points will be considered in formulating a dynamic programming model for an optimal solution of the problem.

Assume that the end of period m and the end of period n are regeneration points for facility 1; that is, $I_{m1}(1) = 0$ and $I_{n1}(1) = 0$. Let A_{mn} denote the cost of producing in period $m+1$ to satisfy demands in periods $m+1, m+2, \dots, n$ ($m=0, 1, 2, \dots, N-1$; $n=m+1, m+2, \dots, N$). A_{mn} includes inventory costs as well as production costs. Then, from Theorem 2,

$$\begin{aligned} A_{mn} &= P_{m+1,1}(x_{m+1,1}) + \sum_{t=m+1}^{n-1} H_{t1}(I_{t1}(1)) + \min \{ P_{m+1,2}(x_{m+1,2}) + \sum_{t=m+1}^{n-1} H_{t2}(I_{t2}), \\ &\min_{2 \leq \lambda \leq n-m} \left[\sum_{t=m+1}^{m+\lambda-1} P_{t2}(x_{t2}) + P_{m+\lambda,2}(x_{m+\lambda,2}) + \sum_{t=m+1}^{m+\lambda-1} H_{t1}(I_{t1}(2)) + \sum_{t=m+\lambda}^{n-1} H_{t2}(I_{t2}) \right] \} \\ &= P_{m+1,1} \left(\sum_{t=m+1}^n r_{t1} \right) + \sum_{t=m+1}^{n-1} H_{t1} \left(\sum_{l=t+1}^n r_{l1} \right) + \min \{ P_{m+1,2} \left(\sum_{l=m+1}^n r_{l2} \right) \\ &\quad + \sum_{t=m+1}^{n-1} H_{t2} \left(\sum_{l=t+1}^n r_{l2} \right), \\ &\min_{2 \leq \lambda \leq n-m} \left[\sum_{t=m+1}^{m+\lambda-1} P_{t2}(r_{t2}) + P_{m+\lambda,2} \left(\sum_{l=m+\lambda}^n r_{l2} \right) \right. \\ &\quad \left. + \sum_{t=m+1}^{m+\lambda-1} H_{t1} \left(\sum_{l=t+1}^n r_{l1} \right) + \sum_{t=m+\lambda}^{n-1} H_{t2} \left(\sum_{l=t+1}^n r_{l2} \right) \right] \}, \end{aligned} \quad \dots(10)$$

where $r_l = r_{l1} + r_{l2}$, $\forall l$.

Let K_n denote the optimal policy costs for periods 1, 2, \dots , n , given $I_{n1}(1) = 0$. Then,

$$F_n = \min_{0 \leq m \leq n-1} (F_m + A_{mn}), \quad n=1, 2, \dots, N, \quad \text{where } F_0 \equiv 0. \quad \dots(11)$$

Similarly, an algorithm for Theorem 3 can be formulated as follows: denoting by B_{mn} the cost of producing in period $m+1$ to satisfy demands in periods $m+1, m+2, \dots, n$ ($m=0, 1, \dots, N-1$; $n=m+1, m+2, \dots, N$),

$$\begin{aligned} B_{mn} &= P_{m+1,1}(x_{m+1,1}) + P_{m+1,2}(x_{m+1,2}) \\ &\quad + \sum_{t=m+1}^{n-1} (H_{t1}(I_{t1}(2)) + H_{t2}(I_{t2})) \\ &= P_{m+1,1} \left(\sum_{l=m+1}^n r_{l1} \right) + P_{m+1,2} \left(\sum_{l=m+1}^n r_{l2} \right) \\ &\quad + \sum_{t=m+1}^{n-1} \left[H_{t1} \left(\sum_{l=t+1}^n r_{l1} \right) + H_{t2} \left(\sum_{l=t+1}^n r_{l2} \right) \right], \end{aligned} \quad \dots(12)$$

and hence the optimal policy costs R_n for periods 1, 2, ..., n , given $I_{n1}(1)=0$, is

$$R_n = \min_{0 \leq m \leq n-1} \{R_m + B_{mn}\}, \quad n=1, \dots, N, \quad \text{where } R_0 \equiv 0. \quad \dots(13)$$

Each of the recurrence relations (11) and (13) is equivalent to that of a shortest path problem. Each relation implies that given any regeneration point n , one can find the optimal last time point prior to n , say $m^*(n)$, when the inventory is to be zero. Therefore, by starting with $n=N$ and working backward, one can identify the regeneration points in the optimal solution.

5. AN EXAMPLE

Production is to be planned for a three-period horizon. There is no initial inventory and the final inventory level is to be zero. No backloggings (shortages) are permitted. Production and inventory costs have the following forms:

$$\begin{aligned} P_{i1}(x_{i1}) &= \begin{cases} s_i + c_i x_{i1}, & \text{if } x_{i1} > 0, \\ 0, & \text{otherwise,} \end{cases} \\ P_{i2}(x_{i2}) &= \begin{cases} z_i + d_i x_{i2}, & \text{if } x_{i2} > 0, \\ 0, & \text{otherwise,} \end{cases} \\ H_{i1}(I_{i1}(1)) &= h_i I_{i1}(1), \\ H_{i1}(I_{i1}(2)) &= \xi_i I_{i1}(2), \text{ and} \\ H_{i2}(I_{i2}) &= \eta_i I_{i2}. \end{aligned}$$

Cost parameters and demands are given in Table 1 :

Table 1. Illustrative Data

Demands and Parameters	Period		
	1	2	3
r_{i1}	4	2	6
r_{i2}	6	3	9
h_i	30	20	40
ξ_i	30	10	20
η_i	20	30	10
c_i	5	4	6
d_i	7	8	6
s_i	30	20	40
z_i	30	20	10

Then, the calculations begin with $n=1$ and Eqs.(10) and (11) are to be used. When $n=1$:
 $A_{01} = P_{11}(r_1) + P_{12}(r_{12}) = 80 + 72 = 152$, and the solution is $x_{11}^* = 10$ and $x_{12}^* = 6$. When $n=2$:

$$\begin{aligned} A_{02} &= P_{11}(r_1 + r_2) + H_{11}(r_{21}) \\ &+ \min \{P_{12}(r_{12} + r_{22}) + H_{12}(r_{22}), P_{12}(r_{12}) + P_{22}(r_{22}) + H_{11}(r_{22})\} \\ &= 165 + \min \{153, 206\} = 318, \text{ and the solution is } x_{11}^* = 15 \text{ and } x_{12}^* = 9; \end{aligned}$$

$A_{12} = P_{21}(r_2) + P_{22}(r_{22}) = 40 + 44 = 84$, and the solution is $x_{21}^* = 5$ and $x_{22}^* = 3$. When $n=3$:

$$\begin{aligned} A_{03} &= P_{11}(r_1 + r_2 + r_3) + H_{11}(r_{21} + r_{31}) + H_{21}(r_{31}) \\ &+ \min \{P_{22}(r_{12} + r_{22} + r_{32}) + H_{12}(r_{22} + r_{32}) + H_{22}(r_{32}), \\ &\min \{P_{12}(r_{12}) + P_{22}(r_{22} + r_{32}) + H_{11}(r_{22} + r_{32}) + H_{22}(r_{32}), \\ &P_{12}(r_{12}) + P_{22}(r_{22}) + P_{32}(r_{32}) + H_{11}(r_{22} + r_{32}) + H_{21}(r_{32})\} \} \end{aligned}$$

$$=540 + \min\{674, \min\{818, 630\}\} = 1170,$$

and the solution is $x_{11}^* = 30$, $x_{12}^* = 6$, $x_{22}^* = 3$,

and $x_{32}^* = 9$;

$$A_{13} = P_{21}(r_2 + r_3) + H_{21}(r_{31}) + \min\{P_{22}(r_{22} + r_{32}) + H_{22}(r_{32}), \\ P_{22}(r_{22}) + P_{32}(r_{32}) + H_{21}(r_{32})\}$$

$$= 220 + \min\{386, 378\} = 598, \text{ and the solution is } x_{21}^* = 20, x_{22}^* = 3, \text{ and } x_{32}^* = 9;$$

$$A_{23} = P_{31}(r_3) + P_{32}(r_{32}) = 130 + 64 = 194, \text{ and the solution is } x_{31}^* = 15 \text{ and } x_{32}^* = 9.$$

These A_{mn} values are now used for computing F_n in Eq. (11) :

$$F_1 = F_0 + A_{01} = 152,$$

$$F_2 = \min\{F_0 + A_{02}, F_1 + A_{12}\} = \min\{318, 236\} = 236, \text{ and}$$

$$F_3 = \min\{F_0 + A_{03}, F_1 + A_{13}, F_2 + A_{23}\} = \min\{1170, 750, 430\} = 430.$$

Thus, the solution of the problem is

$$x_{11}^* = 10, x_{21}^* = 5, x_{31}^* = 15, x_{12}^* = 6, x_{22}^* = 3, \text{ and } x_{32}^* = 9.$$

6. CONCLUSION

Theorem 1 can be directly extended to a multifacility problem with an interrelated demand ratio, say " α to β " ratio; that is, $r_{1u} : r_{1v} = \alpha : \beta$ for $v = u+1 (u=1, 2, \dots)$, where r_{1u} and r_{1v} represent the demands in facility u and facility v , respectively. Accordingly, the algorithms suggested can be easily adjusted.

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