On the Descartes Circle Theorem*

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데카르트의 원정리에 관하여

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=요 약=

본 논문에서는 수학적인 사유형식을 시기적으로 수평과 수직의 軸에서 관찰할 목적으로 Descartes의 원정리을 생각한다. 이 정리에 관해서는 지금까지 접척원의 곡률의 연구가 있으며, 특히 내접원, 외접원의 곡률을 중심적으로 수많은 방법으로 다루어지고 있다. 본 논문에서 이들 방법을 일반화하여 고찰하며 특히 독립적으로 연구되어온 和算의 방법과 비교한다.

1. Introduction

In our previous paper we mentioned that thinking ways of old-time and modern mathematicians are horizontal and vertical, respectively.[7a)]

This is a continuation of the above and we relate the constancy existed in various extensions of the Descartes Circle Theorem and a three dimensional *extension of it. In the latter, particularly,* it is our object to show that the extension corresponds

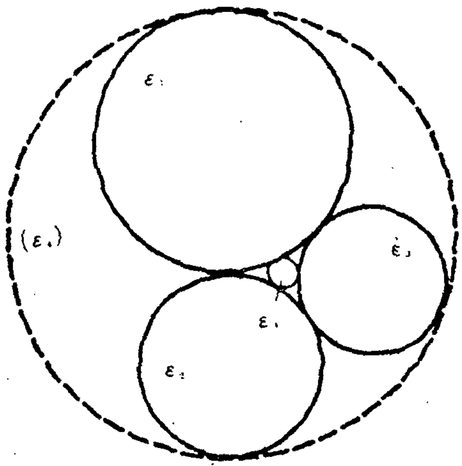

Fig. 1.

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to Sir Frederick Soddy's "The Hexlet".[4)6)]

2. Short History of the Descartes Circle Theorem.

Descartes Circle Theorem, that is a theorem on curvatures of touched circles in Wilker's paper[11)]:

When three circles (curvatures ε_1, ε_2 and ε_3) circumscribe each other and the fourth one (ε_4) contacts the other three, respectively, there is the following relation.

(2.1) $(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4)^2$

$=2(\varepsilon_1^2+\varepsilon_2^2+\varepsilon_3^2+\varepsilon_4^2)$

The proof of (2.1) was simplified by Beecroft.[1)] Steiner solved it on the curvature of the fourth circle, that is,

(2.2) $\varepsilon_4=\varepsilon_1+\varepsilon_2+\varepsilon_3+2\sqrt{\varepsilon_1\varepsilon_2+\varepsilon_2\varepsilon_3+\varepsilon_3\varepsilon_1}$.

Later on Melzak[6)] and Stanton started from Steiner's formula (2.2) and they extended it to n-circles; the former did by n-times repetition of it and the latter by using related difference equation of it.

H.S.M. Coxeter proved it. By using two applications of the Descartes Circle Theorem and he got a linear relation among the bends which gives the result.[2b)] Morley[8)] proved the theorem (2.1) by adopting inversion.

Furthermore Coxeter[2a)] put $\delta=\sqrt{\varepsilon_1\varepsilon_2+\varepsilon_2\varepsilon_3+\varepsilon_3\varepsilon_1}$ and he gave a geometrical interpretation to δ. Supplyng δ to Steiner's formula, it changes into a usable form as follows:

(2.3) $\varepsilon_4=\varepsilon_1+\varepsilon_2+\varepsilon_3+2\delta$.

Then Wilker[11)] extended the above to n-circles. That is, substituting α, β, γ_0 and γ_1 for ε_1, ε_2, ε_3 and ε_4, respectively in Steiner's formula, he defined the n-th circle (curvature γ_n) inductively and he got the following.

(2.4) $\gamma_n=\gamma_0+2n\delta+n^2(\alpha+\beta)$.

3. *The Descartes Circle Theorem and Its Extensions.*

The Descartes Circle Thoerem in "René Descartes' complete works"[3)](R.Descartes, 1596~1650) do not shape itself into a theorem, because it was his letter to princes Elizabeth. In fig. 2, let the letters d, e, f and x be the radius of circles A, B, C and H, respectively. The paragraph of the original is as follows:

Et, en premier lieu, elle trouuera

$$AK\infty\frac{dd+df+dx-fx}{d+f},$$

$$\&\ AD\infty\frac{dd+df+de-fe}{d+f}$$

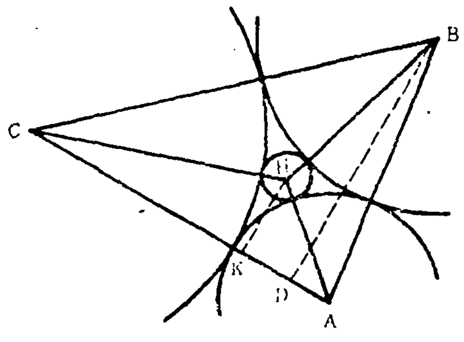

Fig. 2.

oú elle peut desia remarquer que x est dans la ligne AK, comme e dans la ligne AD, pour ce qu'elle se trouue par le triangle AHC, comme l'autre par le triangle ABC. Puis enfin, elle aura cette e'quation,[a)]

(3.1) *ddeeff*∞*2deffxx*+*2deeffx*+ *ddeexx*+*2deefxx*+*2ddeffx*+ *ddffxx*+*2ddefxx*+*2ddeefx*+ *eeffxx*,

de laquelle on tire, pour Theoreme, que les quatre sommes, qui se produisent en multipliant enssmble les quarrez de trois de ces rayons, font le double de six, qui se produisent en multipliant deux de ces rayons l'vn par l'autre, & par les quarrez des deux autres; ce qui suffit pour seruir de regle á trouuer le rayon du plus grand cercle qui puisse estre décrit entre les trois donnez qui s'entretouchent.

Remark 1. The sign∞ is representative of equality=.

Remark 2. The signs of a few terms in the right-hand side(3.1) are misprinted, I think.

By the present time we have not known from what time and by whom the Descartes Circle Theorem had been changed (3.1) to (1.1).

The other hand, Nushizumi Yamaji(山路主住)(1724～1772) wrote "Zeishiki Eendai" 贅式演段 at 1751. In this, he says about the Descartes Circle Theorem as follows[5a)]:

When three circles having radii r_1, r_2 and r_3 circumscribe each other and the fourth one (radius x) touches the other three, there is the following relation:

$$(3.2)\quad (r_1r_2r_3)^2-2r_1r_2r_3(r_1r_2+r_2r_3+r_3r_1)x+(r_1^2r_2^2+r_2^2r_3^2+r_3^2r_1^2-2r_1^2r_2r_3-2r_1r_2^2r_3-2r_1r_2r_3^2)x^2=0$$

Remark 3. In wasan numerical expressions or equations are written vertical writing. Therefore the sign "=" was not used at that time.

Remark 4. Steiner's Theorem is found out in the crelle's Jounal in 1826.

Remark 5. Later on as same as the Decartes Theorem, (3.3) is rewritten by a certain person as follows:

$$(3.3)\quad x^{-1}=s_1'\pm2\sqrt{s_2'},$$

where s_1', s_2' and s_3' are expressed the fundamental symmetric expression of r_1^{-1}, r_2^{-1}, r_3^{-1}.

Theorem 1. Let a point O be the center of two concentric circles having curvatures ρ and 3ρ. Let circles O', O'' (curvature 3ρ) be in contact externally with each other and also with the two circles above. We use the same symbols O', O'' for the centers of these circles.

Let $P_1(\sigma_1)$be the center (curvature) of a circle which is in external contact with the circles O', O'' and the inner circle O.

Moreover, let $Q_1(\tau_1)$ be the center

a) Les, signes+sont omis devant les deux premieres colonnes.

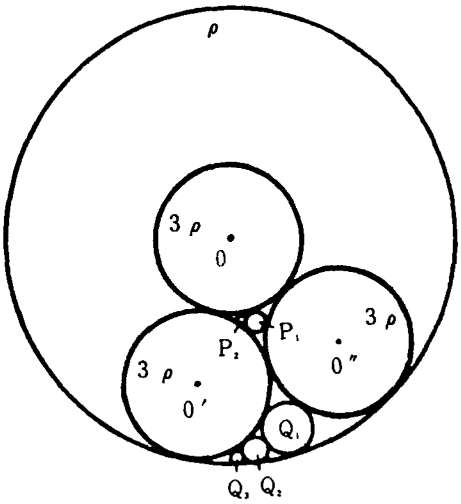

Fig. 3.

(curvature) of a circle which is in external contact with the circles O', O'' and is in internal contact with the outer circle O.

Furthermore, let $\sigma_2(\tau_2)$ and $P_2(Q_2)$ be the curvature and the center of a circle which is in external contact with the circles O', $P_1(Q_1)$and the inner(the outer) circle O,. .. Repeating this processes, we obtain the series $\sigma_2, \sigma_3, \cdots, \sigma_n(\tau_2, \tau_3, \cdots, \tau_n)$ respectively.

Then the following invariable relation is held:

(3.4) $3\tau_n-\sigma_n=6\rho$ $(n=1, 2, \cdots, n)$.

Proof. Applying (2.2) to four circles O', O'', P_1 and O (the inner one), then we have

$\sigma_1=3\rho\times3+2\sqrt{(3\rho)^2\times3}.$

(1) $\therefore\ \sigma_1=(9+6\sqrt{3})\rho.$

On the other hand, applying the Steiner's Theorem to four circles O', O'', Q_1 and O (the outer one), then we get

$\rho=-(3\rho+3\rho+\tau_1)+2\sqrt{3\rho\times\tau_1\times2+(3\rho)^2}.$

(2) $\therefore \tau_1=(5+2\sqrt{3})\rho.$

From (2)×3−(1), then

(3) $3\tau_1-\sigma_1=6\rho.$

Next, we describe the circle P_2(curvature σ_2) which externally contacts with circles O', P_1 and the inner circle O. We obtain the following as the above:

(4) $\sigma_2=(27+12\sqrt{3})\rho.$

Furthermore, we describe the circle Q_2 (curvature τ_2) which externally contacts with circles O', Q_1 and the outer circle O. Then we have

(5) $\tau_2=(11+4\sqrt{3})\rho.$

From (5)×3−(4), then

(6) $3\tau_2-\sigma_2=6\rho.$

Let's consider n circles defined inductively as tangent to the inner(outer) circle fig. 3.:

$P_1, P_2, P_3, \cdots, P_n$ $(Q_1, Q_2, Q_3, \cdots, Q_n)$, and put the curvatures of them $\sigma_i, \tau_i, i=1, 2, \cdots, n$, respectively.

Arrange τ_i as follows:

the value of τ_i	first difference	second difference
$\tau_1=(5+2\sqrt{3})\rho$		
$\tau_2=(11+4\sqrt{3})\rho$	$>(6+2\sqrt{3})\rho$	
$\tau_3=(21+6\sqrt{3})\rho$	$>(10+2\sqrt{3})\rho$	$>4\rho$
$\tau_4=(35+9\sqrt{3})\rho$	$>(14+2\sqrt{3})\rho$	$>4\rho$
⋮	⋮	⋮
⋮	⋮	⋮

The second difference of the sequence $\{\tau_n\}$ is constant and therefore

$$\sum_{i=2}^{n}(\tau_i-\tau_{i-1})=[\{6+10+14+\cdots+(4n-2)\}+2(n-1)\sqrt{3}]\rho.$$

(7) $\therefore \tau_n=(2n^2+2\sqrt{3}n+3)\rho.$

At this point, using (2.2) for four circles O', P_{n-1}, P_n and the inner circle O, we have

(8) $\sigma_n=(6n^2+6\sqrt{3}n+3)\rho.$

Therefore from (7)×3−(8), then

$$3\tau_n-\sigma_n=6\rho.$$

Corollary. Under the assumption of Theorem 1, curvature of one of successive circles is expressed as a form

(3.5) $(x+y\sqrt{3})\rho,$

where x, y are functions of curvature of a circle.

4. An occasion of three dimensions

In our previous paper[7b)], we proved an extension of the problem on Funatsu Jinja's tablet which is as follows:

A large sphere having curvature ρ is packed with the equal 13 spheres having curvature 3ρ.

Let us apply the Descartes circle Theorem to this.

We know the following theorem[5a)].

Theorem 2. There are four spheres touching each other, and all of them circumscribe a large sphere. Let $\varepsilon_i (i=1, 2, \cdots, 5)$ be the curvatures of five sphere, then

$$3(\varepsilon_1^2+\varepsilon_2^2+\varepsilon_3^2+\varepsilon_4^2+\varepsilon_5^2)=(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4+\varepsilon_5)^2.$$

Proof. omitted.

Theorem 3. The large sphere G having curvature ρ is packed with equal spheres having curvature 3ρ.

A_1, A_2, A_3 and A_4 are four spheres of the above equal ones and they touch each other.

Let P_1 be a sphere circumscribed with the above four equal ones, and P_2 be one circumscribed with the spheres A_1, A_2, A_4 and $P_1, \cdots$ Finally P_7 is defined as the same process, then P_7 agrees to P_1.

And more, let Q_1 touch the spheres A_1, A_2, A_4 and large one G, and Q_2 do the spheres A_1, A_2, Q_1 and large one G, $\cdots$. Finally Q_7 is defined as the same process. then Q_7 agrees to Q_1.

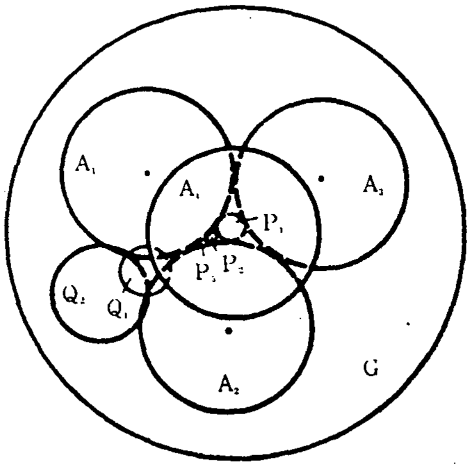

Fig. 4.

Furthermore, let $\sigma_n(\tau_n)$ be the curvature of sphere $P_n(Q_n)$, then

(4.1) $3\tau_n-\sigma_n=6\rho \quad (n=1, 2\cdots, 6).$

Proof. Since the small sphere P_1 touch the spheres A_1, A_2, A_3, A_4 which are circumscribing each other, and the curvatures are given σ_1, 3ρ, 3ρ, 3ρ, 3ρ, respectively, then we obtain

$$(4.2)\quad 3\{(3\rho)^2\times 4+\sigma_1^2\}=(3\rho\times 4+\sigma_1)^2.\ ^{5a)}$$

$$\therefore \sigma_1=(6+3\sqrt{6})\rho,\ (\because \sigma_1>0).$$

Applying the same way to the spheres A_1, A_2, A_4, P_1, P_2, we have

$$3[(3\rho)^2\times 3+\{(6+3\sqrt{6})\rho\}^2+\sigma_2^2]$$

$$=\{3\rho\times 3+(6+3\sqrt{6})\rho+\sigma_2\}^2.$$

Since $\sigma_2>\sigma_1$, then

$$\sigma_2=(12+3\sqrt{6})\rho.$$

We look for suitable curvatures in turn, and then

$$\sigma_3=15\rho,\qquad \sigma_4=(12-3\sqrt{6})\rho,$$

$$\sigma_5=(6-3\sqrt{6})\rho,\quad \sigma_6=3\rho.$$

Therefore, it is evident that $\sigma_7=\sigma_1$.

Moreover, thinking of the spheres A_1, A_2, A_4, Q_1 and large one G, each of whose curvatures is 3ρ, 3ρ, 3ρ, τ and $-\tau$, we use the Theorem 2, then

$$3\{(3\rho)^2\times 3+(-\rho)^2+\tau_1^2\}=(3\rho\times 3-\rho+\tau_1)^2,$$

$$\therefore \tau_1=(4+\sqrt{6}),\ (\because \tau_1>3\rho).$$

For the spheres A_1, A_2, Q_1, G, Q_2, we use the same way as the above, then we have

$$3[(3\rho)^2\times 2+\{(4+\sqrt{6})\rho\}^2+(-\rho)^2+\tau_2^2]$$

$$=\{3\rho\times 2+(4+\sqrt{6})\rho-\rho+\tau_2\}^2.$$

$$\therefore \tau_2=(6+\sqrt{6})\rho,\ (\because \tau_2>\tau_1).$$

Furthermore, we seek suitable curvatures in turn, then

$$\tau_3=7\rho,\qquad \tau_4=(6-\sqrt{6})\rho,$$

$$\tau_5=(4-\sqrt{6})\rho,\qquad \tau_6=3\rho.$$

Therefore, it is evident that $\tau_7=\tau_1$, and

$$3\tau_n-\sigma_n=6\rho,\qquad (n=1,2,\cdots,6).$$

Corollary. Under the assumption of Theorem 3, the curvature of one of successive spheres is expressed as a form

$$(x+\sqrt{6}y)\rho,$$

where x, y are functions of ρ or O.

Remark 6. Relations of σ and τ are as follows:

j \ σ_j, τ_j	σ_j	ε_j
1	$(6+3\sqrt{6})\rho$	$(4+\sqrt{6})\rho$
2	$(12+3\sqrt{6})\rho$	$(6+\sqrt{6})\rho$
3	15ρ	7ρ
4	$(12-3\sqrt{6})\rho$	$(6-\sqrt{6})\rho$
5	$(6-3\sqrt{6})\rho$	$(4-\sqrt{6})\rho$
6	3ρ	3ρ

This table shows that the series $\{P_n\}$ and $\{Q_n\}$ have a period, that is $P_7=P_1$ and $Q_7=Q_1$. The relations of curvatures as follow:

$$\sigma_1<\sigma_2>\sigma_3>\sigma_4>\sigma_5<\sigma_6<\sigma_7=\sigma_1,$$

$$\tau_1<\tau_2>\tau_3>\tau_4>\tau_5<\tau_6<\tau_7=\tau_1.$$

The series $\{Q_i\}$ is expressed as same as a chain of six spheres' theorem: The Hexlet[4)8)] due to Sir F. Soddy.

Using the above table, we have the following relations:

$$\sigma_1+\sigma_4=\sigma_2+\sigma_5=\sigma_3+\sigma_6=18\rho,$$

$$\tau_1+\tau_4=\tau_2+\tau_5=\tau_3+\tau_6=10\rho.$$

Remark 7. On the Hexlet.

Sir F. Soddy(1877～1956) is one of the most famous chemists in Great Britain. He won the Nobel Prize for chemistry in

1921 due to his Isotopic achievement result. His main career is as follows:

1904～1914 Professor University of the Glasgow

1914～1919 Professor University of the Aberdeed

1919～1936 Professor University of the Oxford

The ionic binding is just as same as the chain of six spheres' Theorem. He became aware of Theorem: The Hexlet. We wonder at his ablity to detect this theorem out of his line.

In Japan, many wasan experts have been studied theorems similar to this independently of Soddy. For example[4]:

Gazen Yamamoto 山本賀前(？): Sampō Jo-Jutsu 算法助術 1841

Kagen Fujita 藤田嘉言(1774～1828) : Shinpeki Sampō 神壁算法 I, II (1789)

Gokan Uchida 内田五觀(1805～1882) : Kokon Sankan 古今算鑑(1832)

Shōkō Kenmochi 劍持章行(1790～1871) : Tansaku Sampō 探賾算法(1840)

Kazuhide Ōmura 大村一秀(？～1891) : Sampō Tenzan-Tebiki-so 算法點鼠手引草 2 vol. (1841)

Chōchū Shiraishi 白石長忠(？～1862) : Shamei Sampu 社盟算譜(1826)

8. Conclusion.

In extension of the Descartes Circle Theorem, we have found an universality that what is realized in two dimensions is also done in three dimensions. In three dimensions, this theorem is related to isotopes as "Hexlet Problem".

Through two and three dimensions it is applicable to bearing problem in mechanical and electrical engineering. We believe that our result is valuable for not only mathematics but also engineering.

This paper was based on one of the authors' lecture at Hanyang University and Chungbuk National University in May, 1982 and was rewritten, with giving attention to the process of extension from plane to solid.

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