韓國軍事運營分析學會誌
第10卷, 第2號, 1984.12

A Note on the Minimization of the Expected Makespan and the Expected Flow Time in M Machine Flow Shops with Blocking

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Abstract

Consider an m machine flow shop with blocking. The processing time of job j, j=1, ..., n on each one of the m machines is equal to the same random variable X_j and is distributed according to F_j. We assume that the processing times are stochastically ordered, i.e., $F_1 \underset{st}{\leq} F_2 \underset{st}{\leq} \cdots \underset{st}{\leq} F_n$. We show that the sequence 1,3,5,...,n-1,n,n-2,...6,4,2 when n is even and sequence 1,3,5,...,n-2,n,n-1,...,6,4,2 when n is odd minimizes the expected makespan and that the sequence 1,...,n minimizes the expected flow time.

I. INTRODUCTION

Consider a system of m machines which are set up in series. There are n jobs which are available at time t=0 which have to be processed without preemptions. A job has to be processed first on machine 1, then on machine 2, etc. There is no storage space between any two successive machines. This has the following consequences; a job may start its processing on a machine only if the preceding job has left the machine; a job has to wait on a machine on which it has completed its processing if the preceding job is still occupying the next machine. This phenomenon is called blocking and this kind of system is called a flow shop with blocking. At time t=0 the jobs have to be put in a sequence j_1, ... j_n, a permutation of 1, ..., n, according to which they have to traverse the system. That is job j_1 starts at time t=0 on machine 1; after it has completed its processing there it starts on machine 2, while job j_2 starts on machine 1, etc. The completion time of the last job, i.e., the time the last job leaves the system, is called the makespan and the sum of the completion times of the n jobs is called flow time.

In what follows we assume that the processing times of job j on the m machines are identical and equal to the random variable X_j, which is distributed according to F_j. We also assume that $P(X_i < t) > P(X_j < t)$ for all t when i < j; we say that X_i is stochastically smaller than X_j and write $X_i \underset{st}{<} X_j$

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or $F_i \leq_{st} F_j$. Without loss of generality we may assume that $F_1 \leq_{st} \ldots \leq_{st} F_n$. We are interested in the sequence which minimizes the expected makespan and the sequence which minimizes the expected flow time.

The literature on flow shops with blocking is not extensive. Levner (1969) studied the deterministic flow shop with blocking and suggested a branch and bound algorithm for determining a sequence that minimizes the makespan. Garey et al. (1976) showed that minimizing the flow time in a two machine deterministic flow shop is an NP hard problem. Pinedo (1982) considered a two machine model with blocking in which the processing times of a job on the two machines are random variables which are independent and identically distributed (i,i,d.) and with the distribution functions of any two jobs stochastically ordered. Assuming that $F_1 \leq_{st} \ldots \leq_{st} F_n$ he showed that when n is even the sequence 1, 3, 5, . . ., n-3, n-1, n, n-2, n-4, . . ., 6, 4, 2 minimizes the expected makespan and when n is odd the sequence 1, 3, 5, ..., n-4, n-2, n n-1, n-3, ..., 6, 4, 2 minimizes the expected makespan. The same proof goes through when the processing times of job j on the two machines are equal to the same random variable X_j, which is distributed according to F_j and $F_1 \leq_{st} \ldots \leq_{st} F_n$. Again the sequence 1, 3, 5, ..., n-3, n-1, n, n-2, n-4, ..., 6, 4, 2 when n is even and the sequence 1, 3, 5, ..., n-4, n-2, n, n-1, n-3, ..., 6, 4, 2 when n is odd minimizes the expected makespan. Pinedo also considered an m machine model under the assumption that the processing times of a job on the various machines are identically distributed. He showed that if the densities of any two jobs are nonoverlapping, i.e. $P(X_i \leq X_j)$ is either 0 or 1, any SEPT-LEPT sequence minimizes the expected makespan. A sequence $j_1, \ldots, j_n$ is a SEPT-LEPT sequence if for some k

$$E(X_{j_1}) \leq E(X_{j_2}) \leq \ldots \leq E(X_{j_k})$$

and

$$E(X_{j_k}) \geq E(X_{j_{k+1}}) \geq \ldots \geq E(X_{j_n}).$$

Wie and Pinedo (1984) considered two machine flow shops with blocking in which the processing times of a job on the two machines are not necessarily identically distributed. They presented necessary conditions for sequences to minimize the expected makespan. Foley et al. (1983) showed that the SEPT sequence minimizes the expected flow time in a two machine flow shop with blocking when the processing times of a job on the two machines are i,i,d. and when the distribution functions of any two jobs are stochastically ordered.

In what follows we consider the m machine flow shop with blocking and show that the sequence 1, 3, 5, ..., n-3, n-1, n, n-2, n-4, ..., 6, 4, 2 when n is even and the sequence 1, 3, 5, ..., n-4, n-2, n, n-1, n-3, ..., 6, 4, 2 when n is odd minimizes the expected makespan when the processing times of job j on the m machines are identical and equal to the same random variable X_j. Moreover, we show that the sequence 1, 2, 3, ..., n minimizes the expected flow time. The proofs are based on interchange arguments which are more involved and complicated than the interchange arguments used in the proofs for the two machine models.

The following notation is used.

$C_j(S)$	:	completion time of the jth job under sequence S.
Cmax(S)	:	makespan under sequence S.
ΣC_j (S)	:	flow time under sequence S.
B (j, s)	:	period during which job j occupies machine s.
$\bar{F}(t)$	:	1-F(t) when F is a distribution function.
J_{ik}	:	Sequence $(j_i, j_{i+1}, \ldots, j_{k-1}, j_k)$ if i < k.
		Sequence $(j_i, j_{i-1}, \ldots, j_{k+1}, j_k)$ if i > k.
X V Y	=	max (X, Y)
$X \wedge Y$	=	min (X, Y)
$\lfloor x \rfloor$	:	largest integer less than or equal to x.
1 (j>i)	:	Indicator variable which is 0 when j<i and 1 when j>i.

II. THE RESULTS

Before presenting the main theorems some preliminary results are needed. We first present three lemmas. The first one is not new; it is a special case of a more general result due to Dattatreya (1978) and Muth (1979) that is applicable to our model. We present this (deterministic) result without proof.

Lemma 1

The total time required to process a given sequence of n jobs with processing times $X_1, \ldots, X_n$ is the same as the total time required to process the same n jobs in reverse order.

For our second lemma we assume that $X_{j_1}, \ldots, X_{j_{2k}}$ is a sequence of 2k independent random variables ($j_1, \ldots, j_{2K}$ being a permutation of 1, ..., 2k). We assume that $X_i \underset{-st}{<} X_j$ for all i < j. Let Z denote a random variable that is independent of $X_1, \ldots, X_{2k}$.

Lemma 2

If $v_i = j_i \wedge j_{k+i}$, and $w_i = j_i \vee j_{k+1}$, i=1,2,....,k, then

$$E[(X_{j_1} \vee \ldots \vee X_{j_k} \vee Z) + (Z \vee X_{j_{k+1}} \vee \ldots \vee X_{j_{2k}})] \geqslant$$

$$E[(X_{v_1} \vee \ldots \vee X_{v_k} \vee Z) + (Z \vee X_{w_1} \vee \ldots \vee X_{w_k})].$$

Proof: Let the random variables A, B, C, D be defined as follows:

$$A = X_{j_1} 1(j_1 < j_{k+1}) \vee \ldots \vee X_{j_k} 1(j_k < j_{2k}),$$

and B is defined as A but with the inequalities in the indicator variables reversed.

$$C = X_{j_{k+1}} 1(j_{k+1} < j_1) \vee \ldots \vee X_{j_{2k}} 1(j_{2k} < j_k),$$

and D is defined as C but with the inequalities in the indicator variables reversed. Observe that $A \leq_{st} D$, $C \leq_{st} B$ and A, B, C, D are independent.

Since

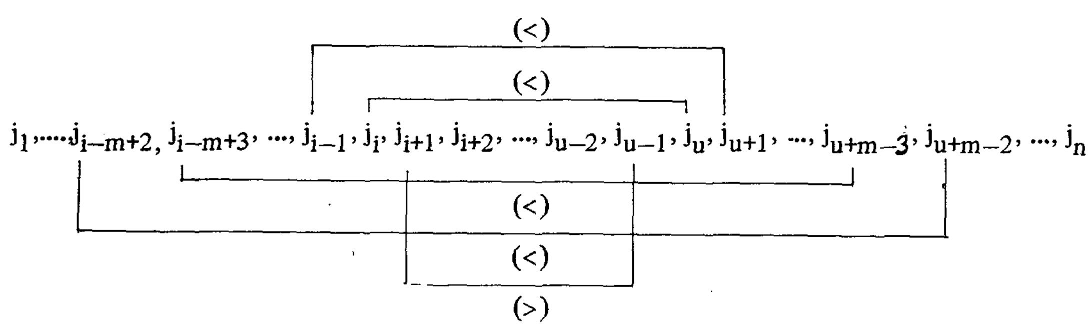
$$P((A \vee Z) \wedge (C \vee Z) > x) = \int_0^\infty P((A \vee Z) \wedge (C \vee Z) > x \mid Z=z)\, dF_Z(z)$$
$$= \int_0^x P(A \wedge C > x)\, dF_Z(z) + \int_x^\infty dF_Z(z)$$
$$= \bar{F}_A(x) \cdot \bar{F}_C(x) \cdot F_Z(x) + \bar{F}_Z(x).$$

It follows that

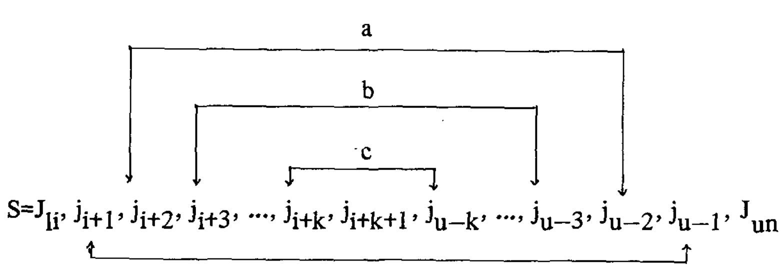
$$E[A \vee B \vee Z + C \vee D \vee Z - A \vee C \vee Z - B \vee D \vee Z] =$$
$$E[-(A \vee Z) \wedge (B \vee Z) - (C \vee Z) \wedge (D \vee Z) + (A \vee Z) \wedge (C \vee Z) + (B \vee Z) \wedge (D \vee Z)]$$
$$= \int_0^\infty F_Z(x)\, [-\bar{F}_A(x) \cdot \bar{F}_B(x) - \bar{F}_C(x) \cdot \bar{F}_D(x) + \bar{F}_A(x) \cdot \bar{F}_C(x) + \bar{F}_B(x) \cdot \bar{F}_D(x)]\, dx$$
$$= \int_0^\infty F_Z(x)\, [\bar{F}_A(x) - \bar{F}_D(x)]\, [\bar{F}_C(x) - \bar{F}_B(x)]\, dx \geq 0.$$

This completes the proof of the lemma.

In the next lemma we compare the makespans under two sequences S and S′. Sequence S has to satisfy some given conditions and sequence S′ is obtained by transforming sequence S in a certain way. Sequence S and the transformation of sequence S into sequence S′ are depicted in Figure 1. We assume that $X_0 = 0$ and that $j_t = 0$ when $t \leq 0$ and $t > n$.



Conditions on Sequence S



$$S' = J_{1i}, j_{u-1}, a(s), b(s), \dots, c(s), j_{i+k+1}, c(\ell), \dots, b(\ell), a(\ell), j_{i+1}, J_{un}$$

$$a(s) = j_{i+2} \wedge j_{u-2} \qquad a(\ell) = j_{i+2} \vee j_{u-2}$$

$$b(s) = j_{i+3} \wedge j_{u-3} \qquad b(\ell) = j_{i+3} \vee j_{u-3}$$

$$c(s) = j_{i+k} \wedge j_{u-k} \qquad c(\ell) = j_{i+k} \vee j_{u-k}$$

Transformation of S into S′

Figure 1

Lemma 3

If S denotes a sequence J_{1n} with $j_{i+1} > j_{u-1}$ and $j_i \leq j_u, j_{i-1} \leq j_{u+1}, \dots, j_{i-m+2} \leq j_{u+m-2}$, for some $1 \leq i < u \leq n$

and if S′ denotes a sequence

$$J'_{1n} = [J_{1i}, j_{u-1}, j_{i+2} \wedge J_{u-2}, \dots, j_{i+k} \wedge j_{u-k}, j_{i+k+1},$$

$$j_{i+k} \vee j_{u-k}, \dots, j_{i+2} \vee j_{u-2}, j_{i+1}, J_{un}]$$

with (u-1-i) odd and $k = \lfloor (u-1-i)/2 \rfloor$, then

$$E[\,Cmax(S)\,] \geq E[\,Cmax(S')\,].$$

When (u–1–i) is even the same statement holds after deleting j_{i+k+1} from sequence S′.

Proof: We prove the lemma under the assumption that (u–1–i) is odd. The same proof goes through for (u–1–i) even. It can be shown (see Pinedo and Weber (1984)) that

$$Cmax(S) = \sum_{t=1}^{n} B(j_t, 1) + \sum_{s=2}^{m} B(j_n, s)$$

$$= \sum_{t=1}^{m-1} (X_{j_1} \vee \dots \vee X_{j_t}) + \sum_{t=m}^{n} (X_{j_{t-m+1}} \vee \dots \vee X_{j_t})$$

$$+ \sum_{t=n-m+2}^{n} (X_{j_t} \vee \dots \vee X_{j_n})$$

Letting $X_{j_t} = 0$ either if $t \leq 0$ or if $t > n$ is equivalent to attaching extra (dummy) jobs with zero processing times at the beginning and at the end. Hence by attaching (m-1) dummy jobs at the end of S $B(j_n, s)$ can be replaced by $B(j_{n+s-1}, 1)$. Now

$$C_{max}(S) - C_{max}(S')$$

$$= \sum_{t=i+1}^{u+m-2} [B(j_t, 1) - B(j'_t, 1)]$$

$$= \sum_{t=i+1}^{i+k+\lfloor m/2 \rfloor} [B(j_t, 1) + B(j_{u+m-2-(t-i-1)}, 1) - B(j'_t, 1) - B(j'_{u+m-2-(t-i-1)}, 1)]$$

$$= \sum_{t=i+1}^{i+k+\lfloor m/2 \rfloor} [(X_{j_{t-m+1}} \vee \dots \vee X_{j_t}) + (X_{j_{u-(t-i)}} \vee \dots \vee X_{j_{u+m-2-(t-i-1)}})$$
$$- (X'_{j_{t-m+1}} \vee \dots \vee X'_{j_t}) - (X'_{j_{u-(t-i)}} \vee \dots \vee X'_{j_{u+m-2-(t-i-1)}})]$$

$$= \sum_{t=1}^{k+\lfloor m/2 \rfloor} [(X_{j_{i+t-(m-1)}} \vee \dots \vee X_{j_{i+t}}) + (X_{j_{u-t}} \vee \dots \vee X_{j_{u-t+(m-1)}})$$
$$- (X'_{j_{i+t-(m-1)}} \vee \dots \vee X'_{j_{i+t}}) - (X'_{j_{u-t}} \vee \dots \vee X'_{j_{u-t+(m-1)}})]$$

If $i+t < u-t$ the above summand is equivalent to a form of Lemma 2 with Z=0. However if $i+t > u-t$ then the quantity $(X_{j_{u-t}} \vee \dots \vee X_{j_{i+t}})$ appear in the four terms and this quantity corresponds to the random variable Z in Lemma 2. It follows from Lemma 2 that $E[Cmax(S)] \geq E[Cmax(S')]$. This completes the proof of Lemma 3.

We are now ready for our first result.

Theorem 1

The sequence 1,3,...,n-2, n,n-1,..., 4, 2 (or its reverse) when n is odd and the sequence 1, 3, ..., n-1, n, n-2, ..., 4, 2 (or its reverse) when n is even, minimize the expected makespan.

Proof: Consider an arbitrary sequcne J_{1n} (say S) and suppose S is $(J_{1,i-1}, 1, J_{i+1,n})$ for some i. Put m-1 extra jobs $-m+1, -m+2, \dots, -2, -1$ with zero processing times in front of the sequence. This addition does not affect the makespan. Since $X_{-m+1} <_{st} X_{j_{i+m-1}}, \dots, X_{-1} <_{st} X_{j_{i+1}}$ and $1 < j_1$ we can apply Lemma 3 and obtain sequence $S' = (1, J'_{2,i-1}, j_1, J_{i+1,n})$ with a smaller expected makespan. After reversing S'and applying Lemma 3 we obtain a sequence $S'' = (2, J''_{1,n-2}, 1)$ again with a smaller expected makespan. Reversing sequence S″ and applying Lemmas 1 and 3 we obtain a sequence $S''' = (1, 3, J'''_{n-3,i+1}, j''_{n-2}, J''_{i-1,1}, 2)$ again with a smaller expected makespan. Continuing in this fashion we obtain one of the sequences stated in the theorem. Them Lemma 1 is applied to complete the proof.

Before stating the next theorem which concerns the sequence that minimizes the expected flow time,. a preliminary result is necessary. In Lemma 4, we compare the expected flow time under two sequences S and S′, in a similar way as in Lemma 3. In Lemma 4 it is assumed that a given sequence has already been improved by putting jobs 1, 2, ..., i-1 in front. It is shown that a further improvement is obtained by scheduling job i after job i-1.

Lemma 4

If S denotes the sequence $J_{1n} = (1, 2, \dots, i-1, J_{i,u-1}, i, J_{u+1,n})$, with $1 \leq i < u \leq n$ and if S′ denotes the sequence $J'_{1n} = (1, 2, \dots, i-1, i, j_{i+1} \wedge j_{u-1}, \dots, j_{i+k} \wedge j_{u-k}, j_{i+k+1}, j_{i+k} \vee j_{u-k}, \dots, j_{i+1} \vee j_{u-1}, j_i, J_{u+1,n})$ with (u-i-l) odd and $k = \lfloor (u-i-1)/2 \rfloor$, then

$E(\Sigma C_i(S)) \geq E(\Sigma C_j(S'))$.

When (u-i-l) is even the same statement holds after deleting job j_{i+k+1} from sequence S'.

Proof: We present the proof only for the case where $n \geq u+m-1$. The case where $n < u+m-1$ is similar.

$$\sum_{t=1}^{n} C_t(S) = \sum_{t=1}^{n} \left[\sum_{s=1}^{t} B(j_s, 1) + \sum_{s=2}^{m} B(j_t, s) \right]$$

$$= \sum_{t=1}^{n} (n-t+1) B(j_t, 1) + \sum_{s=2}^{m} \sum_{t=1}^{n} B(j_t, s)$$

$$\sum_{t=1}^{n} [C_t(S) - C_t(S')] = \sum_{t=1}^{n} (n-t+1) \left[B(j_t, 1) - B(j'_t, 1) \right]$$

$$+ \sum_{s=2}^{m} \sum_{t=1}^{n} \left[B(j_t, s) - B(j'_t, s) \right]$$

Note that the difference between the first summation and the second summation for some s, s=2,..., m, is that each term in the first summation is multiplied by n-t+1. First we prove that the expected value of the first summation is nonnegative. Then we show that it can be deduced from the proof of the fist part that the expected value of the second summation is nonnegative. Since the first and last parts of S and S' are identical, the first summation reduces to

$$\sum_{t=i}^{u+m-1} (n-t+1) \quad B(j_t, 1) - B(J'_t, 1)$$

We partition the above summation into two parts; one part sums from t=i up to $t = i + k + \lfloor m/2 \rfloor$ th other part sums from t=u+m-1 down to $t=u+m-k-\lfloor m/2 \rfloor -1$. Let $a_t = (u+m-1)-(t-i)$. The above can be written as:

$$\sum_{t=i}^{i+k+\lfloor m/2 \rfloor} (n-t+1) \left[B(j_t, 1) - B(j'_t, 1) \right] + \sum_{t=i}^{i+k+\lfloor m/2 \rfloor} (n-(u+m-1)+1+(t-i)) \left[B(j_{u+m-1-(t-i)}, 1) - B(j'_{u+m-1-(t-i)}, 1) \right]$$

$$= \sum_{t=i}^{i+k+\lfloor m/2 \rfloor} (a_t - t) \left[B(j_t, 1) - B(j'_t, 1) \right]$$

$$+ \sum_{t=i}^{i+k+\lfloor m/2 \rfloor} (n-a_t+1) \left[B(j_t, 1) - B(j'_t, 1) + B(j_{a_t}, 1) - B(j'_{a_t}, 1) \right]$$

$$= \sum_{t=i}^{i+k+\lfloor m/2 \rfloor} (a_t - t) \left[(X_{j_{t-m+1}} \vee \ldots \vee X_{j_t}) - (X'_{j_{t-m+1}} \vee \ldots \vee X'_{j_t}) \right]$$

$$+ (n-a_t+1) \left[(X_{j_{t-m+1}} \vee \ldots \vee X_{j_t}) + (X_{j_{a_{t-m+1}}} \vee \ldots \vee X_{j_{a_t}}) - (X'_{j_{t-m+1}} \vee \ldots \vee X'_{j_t}) - (X'_{j_{a_{t-m+1}}} \vee \ldots \vee X'_{j_{a_t}}) \right]$$

It can be shown that $(S_{j_{t-m+1}} \vee \ldots \vee X_{j_t})$ is stochastically larger than $(X'_{j_{t-m+1}} \vee \ldots \vee X'_{j_t})$, for all $t=i, \ldots, (i+k+\lfloor m/2 \rfloor)$ as $A \vee B$ is stochastically larger than $A \vee C$ in Lemma 2. Lemma 2 implies the expected value of the second summand is nonnegative. Hence it follows that $E\,[\sum_{t=1}^{n}(n\text{-}t\text{-}1)\,[\,B\,(J_t, 1)\text{-}B\,(J'_t, 1)\,\}\,] > 0$. If the multiplier (n-t+1) is replaced by 1, the coefficients a_t-t and n-a_t+1 in the above summations have to be replaced by 0 and 1 respectively. It follows again from Lemma 2 that $E\,[\sum_{t=1}^{n} B(J_t, s) - B\,(J'_t, s)\,] \geqslant 0$, for all s=2 , ..., m. This completes the proof of the lemma.

We are now ready for our second result.

Theorem 2

Sequence 1, 2,..., n minimizes the expected flow time.

Proof: The theorem follows by contradiction, since Lemma 4 enables us to improve any sequence which is not according to SEPT.

III. REMARKS

a. We assumed that the processing times of job j on the various machines are identical. If we assume otherwise, for example that the processing times are independent, the explicit expressions of the completion times under a given sequence become more complicated requiring multiple maximum operators.

b. When the size of the intermediate storage is infinite the makespan does not depend on the sequence, but the SEPT sequence does minimize the expected flow time. These facts and the results in this paper indicate that sequences similar to the ones presented in Theorems 1 and 2 may also be optimal when the size of the intermediate storage is positive but finite.

REFERENCES

1. Dattatreya, E., 1978, "Tandem Queueing Systems with Blocking," Ph. D. Dissertation, Industrial Engineering and Operations Research Department, University of California, Berkeley.
2. Foley, R. D. and Suresh, S., 1983, "Stochastically Minimizing the Makespan in Flow Shops," Technical Report VTR8301, Industrial Engineering and Operations Research Department, Virginia Polytechnic Institute and State University, Blacksberg, Va.
3. Garey, M. R. and Johnson, D. S., 1976, "The Complexity of Flow Shop and Job Shop Scheduling," *Math. Oper. Res.*, 1, 117-129.
4. Levner, E. V., 1969, "Optimal Planning of Parts Machining on a Number of Machines," *Automat. Remote Control,* 12, 1972-1981.
5. Muth, E. J., 1979, "The Reversibility Property of Production Lines," *Mgmt. Sci.,* 25, 152-158.
6. Pinedo, M. L., 1982, "Minimizing the Expected Makespan in Stochastic Shop Scheduling," *Opns. Res.,* 30, 148-162.
7. Pinedo, M. L. and Weber, R. R., 1984, "Inequalities and Bounds in Stochastic Shop Scheduling," *SIAM J. of Appl. Math.,* to appear.
8. Reddi, S. S. and Ramamoorthy, C. V., 1972, "On the Flow Shop Sequencing Problem with No Wait in Process," *Opnl. Res. Quart.,* 23, 323-331.
9. Suresh, S., 1984, "Stochastic Flow Shop Scheduling," M. S. Thesis, Industrial Engineering and Operations Research Department, Virginia Polytechnic Institute and State University, Blacksberg, Virginia.
10. Wie, S. H. and Pinedo, M. L., 1984, "On the Minimization of the Expected Makespan in a Two Machine Flow Shop with Blocking," submitted for publication.