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A State-age Dependent Policy for a Shock Process
— Structural Relationships of Optimal Policy —

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Abstract

Consider a failure model for a stochastic system. A shock is any perturbation to the system which causes a random amount of damage to the system. Any of the shocks can cause the system to fail at shock times. The amount of damage at each shock is a function of the sum of the magnitudes of damage caused from all previous shocks. The times between shocks form a sequence of independent and identically distributed random variables. The system must be replaced upon failure at some cost but it also can be replaced before failure at a lower cost. The long term expected cost per unit time criterion is used. Structural relationships of the optimal replacement policy under the appropriate regularity conditions will be developed. And these relationships will provide theoretical background for the algorithm development.

I. INTRODUCTION

We examine a failure model for a system subject to a sequence f randomly occuring shocks. A shock is any perturbation to the system which causes a random amount of damage to the system. Any of

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the shocks can cause the system to fail with failure only possible at shock times. The amount of damage at each shock is a function of the sum of the magnitudes of damage caused from all previous shocks. The times between shocks, called sojourn times, form a sequence of independent and identically distributed random variables. Upon failure, the system is immediately replaced by a new identical system and a failure cost is incurred. If the system is replaced before failure a smaller cost is incurred. In other words, there is an incentive to attempt to replace the system before failure time. The replacement time and replacement cost may depend on the cumulative damage level at replacement time. For any such system in which there is a penalty of some form associated with failure, replacement policies for the system are of considerable interest.

The goal of this paper is to investigate optimal replacement policies using the long-term expected cost per unit time criterion. Previous studies of optimal replacement policies usually were restricted to the class of policies that allowed for replacement only at shock times. In this paper, replacement decisions will be based on both the cumulative damage of the system and the length of time since the last shock. The objective of this paper is to characterize the optimal replacement policy under appropriate regularity conditions. A practical algorithm for finding the optimal policy can be developed through the use of the characteristics found here.

There are two basic methods of approach in deriving optimal replacement policies: the first method is to use semi-Markov decision theory, the second method is to obtain closed form expressions for the optimal policy. Kao (1973) gives the details for using semi-Markov decision theory for the optimal replacement problem on the discrete time parameter space. (In fact, we have taken the terminology "stage-age dependent" policy from Kao.) The advantage in using decision theory lies in its generality; that is, conceptually, it handles problems much more complex than replacement. However, in practice decision theory leads to computational procedures based on cessive approximations which are not always efficient.

Taylor (1975) derives an optimal replacement for a shock process se cumulative damage process is a compound Possion process. In er words, Tayor assumes that the times at which shocks occur form ossion process and the magnitudes of damage caused by each shock m a sequence of independent and identically distributed random iables. The optimal replacement rule in Taylor is a control limit icy and is defined by two simultaneous equations.

Feldman (1976) derives an optimal replacement rule when the ulative damage forms a semi-Markov process. Replacement in Feldman optimal only among the class of control limit policies and is ined implicitly by a single equation. The defining equation was ived using a Markov renewal theoretic approach. While Derman and ks (1960), Kao (1973), and Luss (1976) study the replacement prob- when the state of the system may be observed only periodically, lor (1975) and Feldman (1976) assume that the state of the system be continuously observed.

Bergman (1978) finds replacement policies in situations similar those studied by Taylor and Feldman under the assumptions that proneness to failure of the system is described by an increasing te dependent failure rate function. The optimal replacement rule Bergman is a control limit policy and an algorithm which always es a sequence converging to the optimal level is suggested.

Zuckerman (1978) extends the result of Taylor and Feldman to case where a controller can replace the system at any stopping e before failure. He derives an equation defining the optimal pping time: however, it can be used to find the optimal policy y under very restricted conditions.

Gottlieb (1982) generalizes the above ideas to the case where assumptions about either the monotonicity of failure rate or the sible times of replacement is made, derives the form and proper- s of optimal replacement policy through the Markov decision theo-

retic approach and gives conditions for which a control limit po
is optimal. Fainally, based on the semi-Markov decision theory, Gottlieb provides an algorithm which constructs an optimal repla
ment policy through the semi-Markov decision theory.

The replacement problem is described mathematically in Sect II, and in Section III structural relationships of the optimal p will be developed.

II. STATEMENT OF PROBLEM

Consider a system subject to a sequence of shocks which occ the epochs, T_1, T_2, ..., where the sequence forms a renewal proc with the interrenewal time distribution given by $F(\cdot)$. Damage t system can only occur at shock times and we let Y_t denote the dam to the system accumulated during the interval (o, t). We assume $Y = \{Y_t: t \geqq o\}$ forms a semi-Markov process. The imbeded Markov c which represents the successive deterioration level of the systen will be denoted by $X = \{X_0, X_1, X_2, \ldots\}$, where $X_n = Y_{T_n}$. The tr tion matrix for X is denoted by P with state space E = {1, 1, 2, L}. The state o represents a new system and state L represents a failed system and thus, $P(L,L) = 1$. The potential matrix R of th imbeded Markov chain is defined by

$$R(i,j) = \sum_{n=0}^{\infty} P^n(i,j) \text{ for } i,j \in E$$

and gives the expected number of visits to state j given the chai starts in state i (Cinlar 1975)). At any time t with $T_n \leqq t \leqq T_{n+1}$ assume perfect knowledge of the current state, Y_t, and the last s time, T_n. The failure time, ζ, is defined by

$$\zeta = \inf \{t: Y_t = L\},$$

where $E[\zeta]$ is assumed to be finite. The distribution function F(

and the transition matrix P can be combined giving the semi-Markov kernel $Q = \{Q(i,j,t): i,j \in E, t \geq 0\}$ for the process (X,T) as

$$Q(i,j,t) = \begin{cases} P(i,j)F(t) & i \in E/\{L\},\ j \in E,\ t \geq 0, \\ 1 & i = j = L,\ t \geq 0, \\ 0 & \text{otherwise}. \end{cases} \tag{3}$$

When failures occur, the system must be replaced and a cost $C_1 + C_2$ is incurred. If replacement is made before failure, a cost of C_1 is incurred. It is assumed that when a system is replaced, it is replaced with a new identical system. The criterion to be used for optimization will be the long run expected cost per unit time. A replacement policy is an extended real-valued function, τ, defined on the state space; that is, for each $j \in E$, $0 \leq \tau(j) \leq \infty$. In other words, a replacement policy is a function that defines the maximum time that the process is allowed to spend in a specific state. Let Ξ denote the class of all such policies.

Let U_t be defined as the time elapsed since the last shock of the process Y before t: that is,

$$U_t = t - T_n \quad \text{if} \quad T_n \leq t < T_{n+1}. \tag{4}$$

Then, for any policy $\tau \in \Xi$, the replacement time, S^τ, will be defined by

$$S^\tau = \zeta \wedge \xi^\tau \tag{5}$$

where

$$\xi = \inf \{t: U_t > \tau(Y_t)\} \tag{6}$$

and where the infimum of the empty set is $+\infty$. The notation "$\wedge$" is used to denote the minimum of two functions evaluated pointwise and also used to denote the minimum value between two scalars. It will be necessary to introduce another state to represent a planned replacement and thus the state space is extended accordingly. Let Δ b a distinct point not in E and define E_Δ by

$$E_\Delta = E \cup \{\Delta\}. \qquad (7)$$

For the purpose of evaluating any policy $\tau \in \Xi$, we construct a process (Y^τ, U^τ) which represents (Y, U) under the specified policy Furthermore, we extend the definition of ζ as follows:

$$\xi^\tau = \begin{cases} \zeta \text{ if } \zeta \leq \xi^\tau, \\ \infty \text{ if } \zeta > \xi^\tau. \end{cases} \qquad (8)$$

Then, for a fixed $\tau \in \Xi$ and $t \geqq 0$, we have

$$(Y_t, U_t) = \begin{cases} (U_t, U_t) & \text{if } t < \xi, \\ (\Delta, \infty) & \text{otherwise.} \end{cases} \qquad (9)$$

To define the imbeded Markov renewal process, it is convenient to define the random variable η to designate the first shock after a planned replacement occurs; that is,

$$\eta^\tau = \inf \{n: T_n \geqq \xi^\tau\}. \qquad (10)$$

The imbeded process is

$$(X_n^\tau, T_n^\tau) = \begin{cases} (X_n, T_n) & \text{if } n < \eta^\tau, \\ (\Delta, \infty) & \text{otherwise,} \end{cases} \qquad (11)$$

and, the semi-Markov kernel Q_τ for the process (X^τ, T^τ) is

$$Q_\tau(i,j,t) = \begin{cases} Q(i,j,t \wedge \tau(i)) & i \in E/\{L\}, j \in E, \ t \geqq 0, \\ 1 - \sum_{m \in E} Q(i,m,\tau(i)) & i \in E/\{L\}, \ j=\Delta, \ t \geqq \tau(i), \\ 1 & i \in \{L,\Delta\}, \ j=i, \ t \geqq 0, \\ 0 & \text{otherwise.} \end{cases} \qquad (12)$$

he transition matrix, P_τ, for the imbeded process under the policy is thus given by

$$P_\tau(i,j) = \begin{cases} P(i,j)\ F(\tau(i)) & i\in E/\{L\},\ j\in E, \\ 1 - F(\tau(i)) & i\in E/\{L\},\ j=\Delta, \\ 1 & i\in\{L,\Delta\},\ j=i, \\ 0 & \text{otherwise}. \end{cases} \qquad (13)$$

: will be necessary to use the potential matrix, R_τ, for the Markov ain X^τ; that is, define R_τ, for $i,j\in E$, by

$$R_\tau(i,j) = \sum_{n=0}^{\infty} P_\tau^n(i,j) \qquad (14)$$

ere P_τ^n denotes the nth power of the matrix P_τ. For the simplicity notation, we always assume that the system starts at time 0 in ate 0; that is, $X_0^\tau = 0$ and $T_0^\tau = 0$, for all $\tau\in\Xi$.

Our goal now is to give the long run average cost associated th a specific replacement policy. This is accomplished by forming renewal process describing successive replacements and then using andard results for renewal reward process.

Let S^τ be the random variable denoting the replacement time terval under the policy τ (Equation (5)). Since the system is ways replaced with a new system at each replacement time, the future olution of the process after replacement is always independent of s past behavior and always governed by the same probability law. us, we have an infinite number of replacement cycles which are in-pendent and identically distributed. Let V_1^τ, V_2^τ, ... be the succes-ve replacement epochs and S_0^τ , S_1^τ, S_2^τ, ... be the successive replace-nt cycle times. In other words, $V_0^\tau = S_0^\tau = 0$ and, for $n\geq 1$, $v_n^\tau = V_{n-1}^\tau$ S_n^τ. Also, let N_t be the number of replacement cycles in $(0, t)$. cause of the independence and identical assumption for replacements, e process $V^\tau = \{V_0^\tau, V_1^\tau, \ldots\}$ is a renewal process and $\{N_t,\ t\geq 0\}$ is its associated counting process. Let C_1^τ, C_2^τ, ... be the independen identically distributed random variables representing successive re placement cycle costs under the policy where C_1^τ is defined by

$$C_1^\tau = \begin{cases} C_1 & \text{if } S_1^\tau = \xi^\tau, \\ C_1 + C_2 & \text{if } S_1^\tau = \zeta^\tau. \end{cases}$$

Let $\psi(\tau)$ denote the long run expected cost per unit time under the policy τ; that is, we have

$$\psi(\tau) = \lim_{t \to \infty} E\,[C_1^\tau + C_2^\tau + \ldots + C_{N_t}^\tau] / t$$

Then by the renewal reward theorem of Ross (1970b, Theorem 3.16), it follows that

$$\psi(\tau) = E[C_1^\tau]/E[S_1^\tau]. \quad (15)$$

Since $E[C_1] = C_1 + C_2 \Pr\{S_1^\tau = \zeta^\tau\}$, we have the following equation:

$$\psi(\tau) = \frac{C_1 + C_2 \Pr\{S_1^\tau = \zeta^\tau\}}{E[S_1^\tau]}.$$

Let $\mu_\tau(j)$ be the expected sojourn time at the state j under the policy τ. Then, for $j \in E/\{L\}$

$$\mu_\tau(j) = \int_0^{\tau(j)} [1 - F(t)]dt. \quad (16)$$

It should be noted that for every policy $\tau \in \Xi$, the decision at state L will be limited to $\tau(L) = 0$ to force immediate replacement upon failure.

The standard Markov renewal arguments can now be used to obtain an expression for the long run average cost in a similar manner to the derivations in Feldman (1976). Thus, under the policy τ, the long run average cost is

$$\psi(\tau) = \frac{C_1 + C_2 \sum_{j=0}^{L-1} R_\tau(o.j)\ P_\tau(j,L)}{\sum_{j=0}^{L-1} R_\tau(o,j)\mu_\tau(j)} \tag{17}$$

The optimization problem for replacement is to find the function such that

$$\psi(\tau^*) = \inf \ \{\psi(\tau) : \tau \in \Xi\} \tag{18}$$

ere $\psi(\tau)$ is defined by Equation (17).

III. STRUCTURAL RELATIONSHIPS OF OPTIMAL POLICY

In this section basic structural characteristics are developed ich can be used to find an optimal replacement policy defined by uation (18). In the process of characteristic development, the st function defined by Equation (17) will be fully exploited under xt three regularity conditions.

It is intuitively reasonable that replacement policies are con- dered for equipment that undergoes "wear-out"; that is, for equip- nt that deteriorates as it ages. This concept is made precise in e increasing failure rate (IFR) definitions for Markov chains and obability distribution functions.

FINITION (3.1) (Barlow et al. (1965)). A Markov chain with state ace E is said to be IFR if Pr $\{X_1 \geq k \mid X_0 = i\}$ is nondecreasing i for every $k \in E$.

other words, if a Markov chain is IFR, $q_k(i) = \sum_{j=k}^{L} P(i,j)$ is a non- creasing function in i for all $k \in E$. An IFR Markov chain is a chain which the probability of "becoming worse" at the next jump increases as the initial state increases. (This is equivalent to condition I in Derman (1963) and also condition 3 in Kao (1973).)

DEFINITION (3.2) For a given probability function $F(\cdot)$ with probability density function $f(\cdot)$, its associated hazard rate function $h(\cdot)$ is defined, for $t \geqq 0$, by

$$h(t) = f(t)/[1 - F(t)].$$

DEFINITION (3.3) A probability distribution function F is said to be IFR if $\bar{F}(x \mid t) = \bar{F}(x + t)/\bar{F}(t)$ is decreasing in $-\infty < t < \infty$ for eac $x \geqq 0$ where $\bar{F}(x) = 1 - F(x)$.

Definition (3.3) does not require the existence of a density function. However, when the density exists, Definition (3.3) is eq valent to saying that F is IFR if its hazard rate, $h(\cdot)$ is increasi Now, we are ready to state the necessary regularity conditions.

REGULARITY CONDITIONS (3.4) Let (X,T) be the Markov renewel proces defined by Equation (3). The process (X,T) is assumed to have the following properties:

(a) The sojourn time distribution function F is IFR and has a hazard rate function denoted by $h(\cdot)$.

(b) Imbeded Markov chain is IFR.

(c) Each state in E is accessible from state 0.

The conditions (a) and (b) refer to the previous definitions. Cond tion (c) is equivalent to $R(o,m) > 0$ for all $m \in E$ where R is define by Equation (1). This final condition causes no loss of generality since if $R(o,m) = o$ for some m, then the state space could simply b redefined excluding state m without affecting the probability law o the process.

We now return to the cost function $\psi(\cdot)$ defined by Equation (1 and derive some of its properties under the Regularity Conditions (3.4).

Throughout this section, the superscript "*" denotes an optimum: for example τ^* represents an optimal policy for a given problem and $\tau^*(j)$ denotes the optimal policy at state $j \in E$. (It should be remembered that $\tau^*(L) \equiv o$.) In addition to the regularity conditions, we always assume that $F(o) < 1$, $C_1 > 0$, and $C_2 > 0$.

We begin by obtaining one of the fundamental functional relationships.

LEMMA (3.5) Consider ψ as a real valued function defined by Equation (17). Let $\tau \in \Xi$ and $m \in E$ such that $R_\tau(o,m) \neq 0$, then the condition

$$\partial\psi(\tau)/\partial\tau(m) = 0$$

is equivalent to

$$\psi(\tau) = H_m(\tau(m), \tau(m+1), \ldots, \tau(L)), \tag{19}$$

where H_m is defined by

$$H_m(\tau(m), \ldots, \tau(L)) = \frac{C_2 \cdot \theta_\tau(m,L)}{[1/h(\tau(m))] + \sum_{j=m+1}^{L-1} \theta_\tau(m,j) \cdot \mu_\tau(j)} \tag{20}$$

with

$$\theta_\tau(m,j) = \begin{cases} o & j \leq m, \\ p(m,j) & j = m+1, \\ p(m,j) + \sum_{i=m+1}^{k-1} \theta_\tau(m,i) P_\tau(i,j) & j > m+2. \end{cases} \tag{21}$$

Furthermore,

$$\partial\psi(\tau)/\partial\tau(m) < 0$$

is equivalent to

$$\psi(\tau) > H_m(\tau(m), \ldots, \tau(L)) \tag{22}$$

and

$$\partial\psi(\tau)/\partial\tau > 0$$

is equivalent to

$$\psi(\tau) < H_m(\tau(m), \ldots, \tau(L)). \tag{23}$$

Proof, see Joo (1983) for the proof.

The function H_m defined by Equation (20) will play a key role ir the proofs to follow. The value of $\theta_\tau(m,j)$ can be interpreted as the probability of eventually reaching state j from state m given that policy τ is followed. One of the main advantages of Lemma (3.5) is the fact that θ_τ is a function of $\tau(j)$ for $j \geq m+1$ and thus the depend ence of H_m on τ_m is only through the hazard rate function, $h(\cdot)$.

LEMMA (3.6) Let H_m be the function defined by Equation (20). Then H_m is an increasing function of $\tau(m)$.

Proof. It follows from the Regularity Condition (3.4b) and from Equa tion (20) since θ_τ is independent of $\tau(m)$.

In the following analysis, it will be convenient to investigate ψ along one coordinate. Thus, for a fixed $\tau \in \Xi$, we let ψ_m denote th cost associated with τ as a function of $\tau(m)$ only; that is,

$$\psi_m(t) = \psi(\tau(0), \ldots, \tau(m-1), t, \tau(m+1), \ldots, \tau(L)),$$

where the dependence of ψ_m on τ is not explicitly shown for ease of notation. Three lemmas are now presneted which provide characterist of the structure of ψ_m.

LEMMA (3.7) Consider a fixed policy τ Ξ and a nonnegative integer $m < L$. Under the Regularity Conditions (3.4), the function $\psi_m(\cdot)$ has at most one finite valued minimum if $R_\tau(o,m) \neq 0$.

roof. Assume otherwise; that is, assume ψ_m has a local minimum at $_1$ and t_3. Then, there must be a value, t_2, such that $t_1 < t_2 < t_3$ ınd t_2 is a local maximum. Then by Lemma (3.5) and (3.6), we have

$$\psi_m(t_2) = H_m(t_2, \tau(m+1), \ldots, \tau(L))$$

$$< H_m(t_3, \tau(m+1), \ldots, \tau(L)) = \psi_m(t_3)$$

This leads to contradiction since the assumption was that $\psi_m(t_3) <$ $_m(t_2)$.

.EMMA (3.8) Consider a fixed policy $\tau \in \Xi$ and a nonnegative integer $\imath < L$. Under the Regularity Conditions (3.4), $\psi_m(\cdot)$ can not have a local maximum occuring at a point that is strictly positive and finite if $R_\tau(o,m) \neq 0$.

Proof. Assume otherwise, and let t^1 denote a local maximum. Then there exists $t^2 > t^1$ such that

$$\psi_m(t^2) < \psi_m(t^1) \text{ and } d\psi_m\ (t^2)/dt < 0.$$

But we know by Lemma (3.5) and (3.6) that

$$\psi_m(t^2) \geq H_m(t^2, \tau(m+1), \ldots, \tau(L)) > H_m(t^1, \tau(m+1), \ldots, \tau(L)) = \psi_m(t^1)$$

which is a contradiction.

The above two lemmas indicate that $\psi(\cdot)$ under the Regularity Conditions (3.4) is unimodal along any coordinate direction m if $R_\tau\ (o,m) \neq 0$. However, if $R_\tau\ (o,m) = 0$, the following lemma is useful.

LEMMA (3.9) Consider a fixed policy $\tau \in \Xi$ and a nonnegative integer $m < L$. Under the Regularity Conditions (3.4), $\psi_m(\cdot)$ is constant if $R_\tau\ (o,m) = 0$.

Proof. We know that $R_\tau(o,o), \ldots, R_\tau(o, m-1), R_\tau(o, m+1), \ldots, R_\tau(o,L)$ are independent of $\tau(m)$ if $R_\tau(o,m) = 0$ and also from Equation (17) that

$\psi(\tau)$ is independent of $\tau(m)$. Therefore we have the desired result.

Combining the above three lemmas, we have shown that the graph of $\psi_m(\cdot)$ will take one of the four forms depicted in Figure 1.

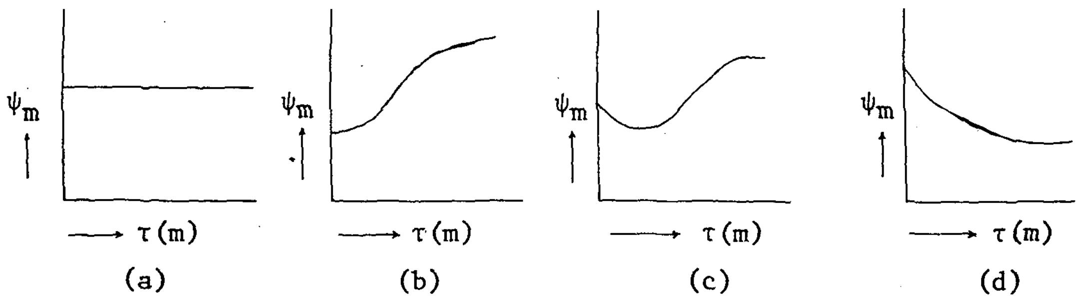

Figure 1. The graph of ψ_m

Noting that case (a) of Figure 1 can be occured only when $R_\tau(o,m)=0$ we know that every point $\tau(m) \in [0,\infty]$ can be regarded as an optimal policy at state m. However, it will be shown that if $f(0) = 0$, the $R_\tau(o,m) > 0$ for all $m \in E$ and thus Case (a) of Figure 1 will not be relevant. We first establish an ordering of the optimal policy usi a result of Gottlieb (1982). Then it will be shown that the optima policy must always be strictly positive.

LEMMA (3.10) Let τ^* be the optimal policy as defined by (18) under the Regularity Conditions (3.4). Then $\tau^*(0) \geq \tau^*(1) \geqq \ldots \geqq \tau^*(L)$.

Proof. This is Theorem 3.4 of Gottlieb. His conditions are that tl hazard rate be nondecreasing and that the probability of a function· ing system surviving the next shock to be nonincreasing as the stat increases. These properties are satisfied by the Regularity Conditions (3.4 a,b).

The positivity of τ^* is shown in the following lemma for the case when the density function of the intershock times evaluated at zero is zero.

.EMMA (3.11) Let τ^* be the optimal policy as defined by Equation (18) ınder the Regularity Conditions (3.4). If $f(o) = 0$, then $\tau^*(k) > 0$ for ıll $k < L$.

'roof. See Joo (1983) for the proof.

:he following theorem summarizes the previous lemmas regarding the ›ptimal replacement policies.

'HEOREM (3.12) Let τ^* be the optimal policy as defined by Equation (18) under the Regularity Conditions (3.4) and let $\psi^* = \psi(\tau^*)$. Then :or each nonnegative $m < L$, we have (possibly both)

$$\tau^*(m) = 0, \text{ or}$$

$$H_m(\tau^*(m), \ldots, \tau^*(L)) = \psi^* \qquad (26)$$

'roof. See Joo (1983) for the proof.

Equation (26) will become a key property in developing a practi- :al algorithm especially when used together with an ordering relation- ıhip on the optimal replacement times. By combining Lemma (3.10) and :heorem (3.12) it is easy to show the following lemma.

.EMMA (3.13) Let τ^* be the optimum policy as defined by Equation (18) ınder the Regularity Conditions (3.4) and let m be a positive integer .ess than L. If $\tau^*(m+1) \neq 0$, then we have

$$H_m(\tau^*(m), \ldots, \tau^*(L)) = H_{m+1}(\tau^*(m+1), \ldots, \tau^*(L)) \qquad (27)$$

'roof. Since $\tau^*(j) \neq 0$ for $j \leq m+1$, we know that $R_\tau(o, m+1) \neq 0$ by :he Regularity Condition (3.4 c) and thus Lemma (3.10) can be used. 'rom Lemma (3.10), if $\tau^*(m+1) \neq 0$ then $\tau^*(m) \neq 0$ and Equation (26) ıost hold and the result follows.

Lemma (3.13) indicates that $\{\tau^*(0), \tau^*(1), \ldots, \tau^*(k-1)$ can be obtained through Equation (27) if once we have $\tau^* (k) > 0$.

IV. CONCLUSION

In section III we developed a structural relationships for the practical algorithm development.

The properties in Lemma (3.10) and Lemma (3.13) will play a sig ificant role in the development of new algorithm.

Thus the algorithm based on the structural relationships developed here need not depend on the policy iteration method of Markov Decision theory which is often inefficient. New algorithm developme based on this Markov renewal theory is required.

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