〈研究論文〉

Journal of the Korean Institute of Industrial Engineers
Vol. 9. No. 1, June 1983

A GENERAL METHODOLOGY FOR ESTABLISHING OPTIMAL INSPECTION POLICY IN A COMPLEX SYSTEM

SOUNG HIE KIM*

Abstract

This paper develops a general methodology for evaluating inspection (mainly, safety inspection) policies, standard and regulations in a complex system with lots of components. Based on practically available data, this methodology enable planners and regulators to estimate the costs and effectiveness of different inspection policies when applied under different system working conditions. As main tools, the concepts of detection probability and earliness of detection are developed in this paper.

1. Introduction

In this paper we analyze an inspection program with two tests for a single defect in a complex system.

Once a defect (e.g., a crack, a melted bearing, or an unbalance load) occurs in a system, it gets progressively worse until it either causes the system to fail, or is detected and repaired or replaced. How far the defect has progressed from the time of its genesis is called **the stage of the defect**. The purpose of testing is to detect defects in an early stage, before failure and at a time when repair costs are low.

The history of a system is marked by certain **events**. Examples are the event that a defect first develops, the event that a test detects a defect, and the event that the system fails. A **decision point** in the system's history at which we much choose which tests, repairs, or maintenance actions to perform, if any.

2. Modelling the Problem

The structure of the problem is indicated in Figure 1. In the following discussion, the letters in parentheses refer to the events depicted in Figure 1.

Consider a system (a) in an OK state (all previous tests have been negative). At this point, there is the option of performing Test A(b), per-

*KAIST

forming Test B(c), performing routine maintenance (d), or doing nothing (e). Suppose the system is given Test A. The test might result positive (f) (indicating the possibility of a serious defect), or negative(g). If the test results positive the system should be stopped for a more complete inspection. A positive test result may be incorrect (a false-positive test result) (h), or the test result may be correct, having detected a serious defect (a true-positive test result) (i). Such a true-positive test should be followed by repair of the component (j), before the system is returned to work. As the system works through the interval, it either fails (k) or does not fail (l). If the system receives a negative test, it also either fails (m) or does not fail (n) during the interval before next inspection.

Suppose that instead of Test A, the system is given Test B (c). A positive test (o) indicates the need for repair (p), whereas a negative test (q) rules out the need for such service. (A false-positive result is also possible, but is omitted to simplify the decision.) As before, following complete inspection and repair, the system works again through the interval before the next inspection. In that interval it can either fail (r) or not fail (s). The same outcomes (t) and (u), are possible for the system that has successfully passed Test B without a detected defect.

The system can be given routine maintenance (d) before proceeding through the next interval. After receiving routine maintenance, the system either fails (v) or does not fail (w) before the next inspection.

A final option is that the system passes with no tests (or services). As in the other cases, the system either fails (x) or does not fail (y) before the next inspection.

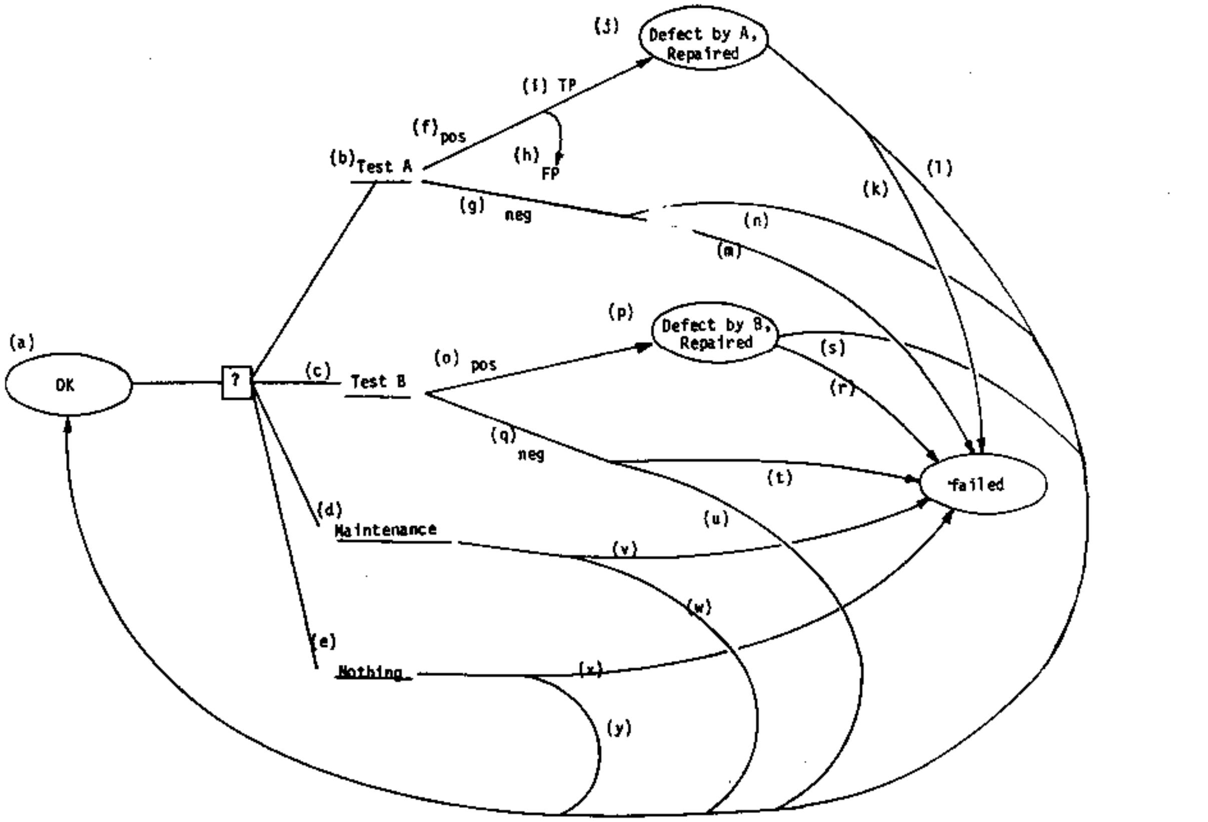

Fig. 1. Structure of the problem.

The structure in Figure 1 suggests a stochastic model that traces the expected fate of a system under various inspection policies. The structure can be transformed into the state transition diagram in a time-varying, state-varying discrete time Markov chain (1). The term state - varying is introduced because the states of the Markov chain change slightly with each transition to take into account changes in the age and screening history of the system. A state is denoted with the letter "S" and two subscripts. The first subscript designates the system's condition and the second represents the time defining the system's condition. For example, if a system at time t_0 is OK, the system is in State S_1, t_0.

Now consider the system at inspection time t_0. The system is in the State S_1, t_0 as shown in Fig. 2. The transitions out of S_1, t_0 depend on the inspection policy. The policy designates whether Test A, Test B, routine maintenance, or no action is to be performed at t_0. For example, if the policy stipulates that only Test A is to be performed at t_0, then the state transition diagram becomes that shown in Figure 2.

Suppose that at t_1, the inspection policy calls for the performance of Test B. The state transition diagram looks like Figure 3, and the transition probabilities are conditional on the current state definitions. For example, the probability of moving from S_1, t_1 to S_3, t_2 is the probability that a system with previous history plus $(t_1 - t_0)$ age and which did not fail through the t_1, has a visually observable defect (Test B). At t_2, then the "OK" condition S_1, t_1 is defined to contain systems with previous history plus $(t_2 - t_0)$ age, had a negative Test A $t_2 - t_1$ hours ago, and a negative Test B $t_2 - t_1$ hours ago.

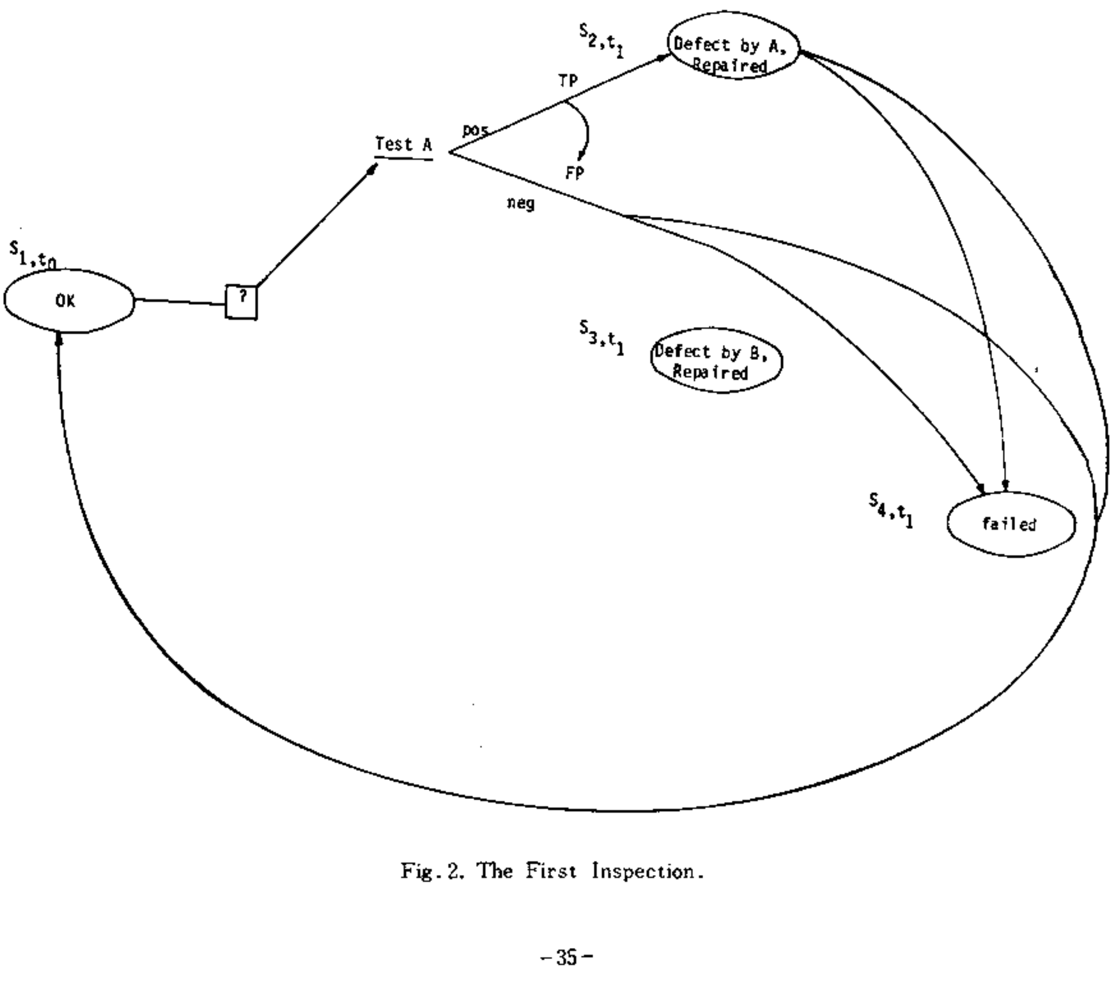

Fig. 2. The First Inspection.

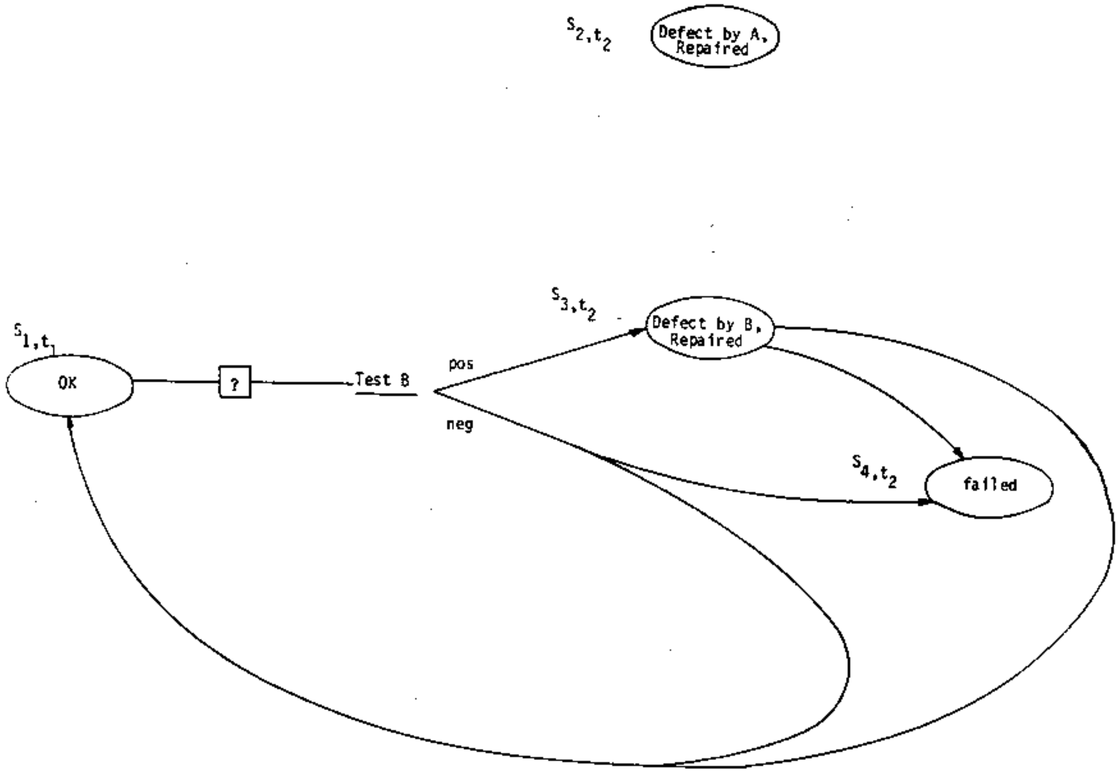

Fig. 3. The Second Inspection Defect by A, Repaired

In this way, the structure of the model unfolds as the calculations proceed. At each interval, the transition probabilities are computed as a function of the current state definitions, and only those state, transitions, and transition probabilities needed to model a particular scenario are generated.

In order to calculate the effect of any inspection interval, the step size of the Markov chain can be left as a variable that is determined by the inspection policy.

3. Basic Formulas of the Model

Many formulas are used to compute the transition probabilities used in the model. The letters in parentheses refer to events illustrated in Figure 1.

1) The probability that at any time Test A detects a defect (f and i) in an OK system.
2) The probability that at any time Test B detects a defect (o) in an OK system.
3) A probability density function on earliness of detection, given each way (b or c) that the defect can be detected. The earliness of detection affects the probability of failure during the subsequent interval (k and r).
4) Given that Test A or Test B was negative at the last inspection, the probability that the system fails before next inspection (m and l), and
5) Given that no actions were performed, the probability that the system will fail before next inspection (x).

Each of these probabilities is a function of at least the following:

1) The time previously worked.
2) The risk of developing the defect compared to the average systems of the same type. This risk may depend on such factors as the working load, quality of routine servicing,

and so forth.

3) The interval from the last application of Test A.

4) The interval from the last application of Test B.

5) The accuracy of the tests.

With these probabilities, we can obtain all the transition probabilities required to define a time-varying, state-varying Markov model that describes the impact of inspection policy.

To derive these formulas, we define two parameters, the "A-interval", α, and the "B-interval", β, where the letters refer to the titles of the tests. These parameters can be visualized in Figure 4, which represents the development of a defect (for instance, crack) in a component of a system. The vertical and horizontal axes represent the stage of development of the crack and the number of age worked by the component, respectively. The more system worked, the worse the crack.

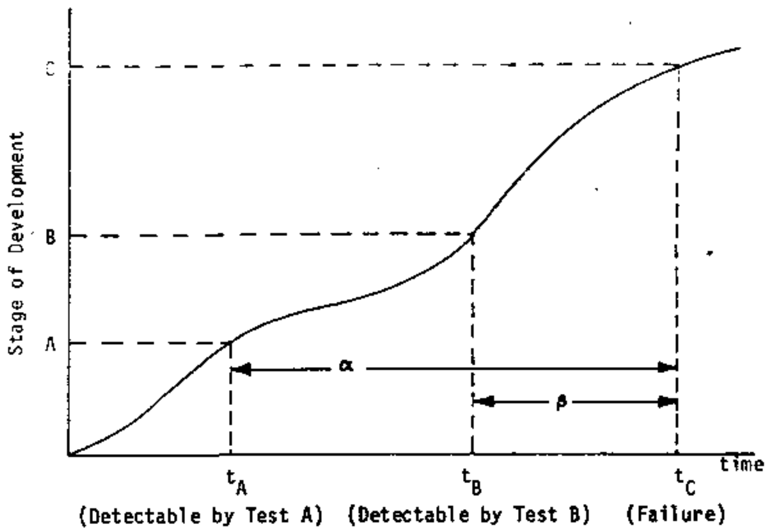

Fig. 4. Development of Defective Component.

Each cracked component progresses at its own rate. Thus each component will have a particular A-interval and a particular B-interval. In a large group of systems, there will be a range of A-intervals and B-intervals depending on the kinds of systems, the condition of loading, the accuracy of testing equipment, the conscientiousness of inspectors, and many other factors. Thus α and β are random variables. Define $f(\alpha)$ and $g(\beta)$ as the probability density functions, and $F(\alpha)$, $G(\beta)$ as the cumulative distributions of α and β respectively.

All of the formulas needed to write the Markov model can be derived in terms of these intervals. In that model we make two assumptions: (1) α and β are independent, and (2) if once a defect is detectable by a kind of test it is always detectable by that kind of test. The first assumption simplifies the analysis and the theory can be extended to the case in which α and β are dependent. The second assumption is called the Progression Assamption In the cases of defects that as cracks, wear and tear, the Progression assump progress continuously, such tions appears to be a good one.

4. Derivation

Let us derive a formula for the probability that at any time, Test A, if performed, discovers a defect.

The related events are defined below and diagrammed in Figure 5.

t_0 is the current time.

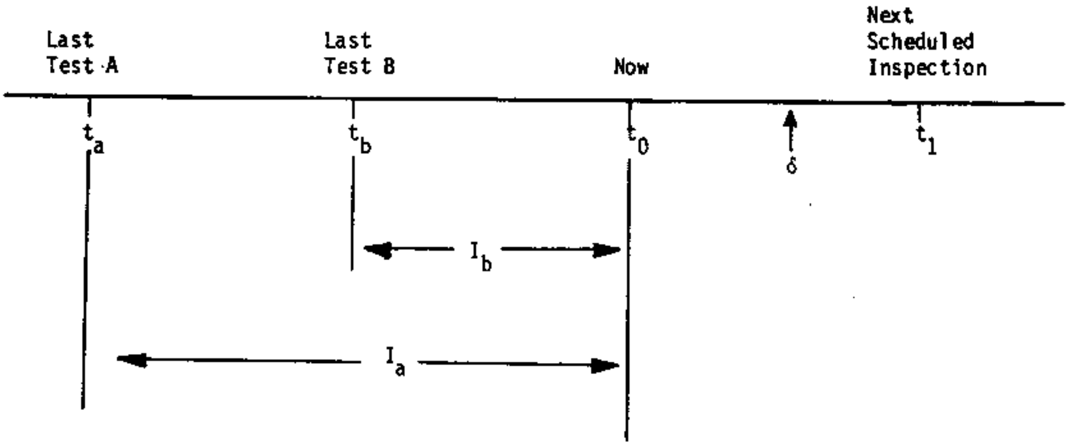

Fig. 5. History of system.

t_a is the system was last inspected using Test A.

t_b is the time the system was last inspected using Test B.

t_1 is the time of the next scheduled inspection.

A_a^- is the event that Test A performed at t_a was negative.

B_b^- is the event that Test B at t_b was negative.

C_0^- is the event that the system has not failed (is OK) at time t_0.

δ is the actual time the defect causes a failure in the absence of any inspections. Thus δ is a random variable, and corresponds to t_c in the history of a defect (Figure 4)

Let $\{O\}$ denote the probability of event O. Let $\{A \mid R\}$ denote the conditional probability of event A, given event R. Using this notation, the probability that Test A done now (at t_0) will detect a defect (event A_0^+) given that the system was not failed (C_0^-) and that Test A performed at t_a and Test B done at t_b are both negative (A_a^-, B_b^-), can be written as $\{A_0^+ \mid A_a^-, B_b^-, C_0^-\}$. We now derive a formula for this probability.

First, expand over δ to get

$$\{A_0^+ \mid A_a^-, B_b^-, C_0^-\} = \int_\delta \{A_0^+, \delta \mid A_a^-, B_b^-, C_0^-\} = \int_\delta \{A_0^+ \mid \delta, A_a^-, B_b^-, C_0^-\} \{\delta \mid A_a^-, B_b^-, C_0^-\}. \quad (1)$$

If we assume the Progression assumption, then the left half of the integrand becomes

$$\{A_0^+ \mid \delta, A_a^-, B_b^-, C_0^-\} = 1 - F(\delta - t_0) \quad (2)$$

since, given a particular δ, Test A done at t_0 will be positive (A_0^+) if and only if the A-interval is greater than $\delta - t_0$. In anthropomorphic terms, $1 - F(\delta - t)$ is the probability that Test A can look far enough into the future to detect a defect that will be obvious $\delta - t_0$ years from now.

Using Bayes' formula we can write the second half of Eq. (1) as

$$\{\delta \mid A_a^-, B_b^-, C_0^-\} = \frac{\{A_a^-, B_b^- \mid \delta, C_0^-\} \{\delta \mid C_0^-\}}{\int_\sigma \{A_a^-, B_b^- \mid \delta, C_0^-\} \{\delta \mid C_0^-\}} \quad (3)$$

Consider now the first probability in the numerator of Eq. (3) $\{A_a^-, B_b^- \mid \delta, c_0^-\}$. Given δ, we want the probability that (1) Test A at t_a was negative (A_a^-), and (2) Test B at t_b was negative, (B_b^-). For a particular δ, Test A will have been negative at t_a if the A-interval is shorter than $\delta - t_a$. This is $F(\delta - t_a)$, the probability that the A-interval is short enough so that a defect that causes a failure at time δ is not detectable by Test A when done at t_a. Also, given δ, Test B will have been negative if the B-interval is shorter than $\delta - t_b$, which occurs with probability $G(\delta - t_b)$. Hence

$$\{A_a^-, B_b^- \mid \delta, C_0^-\} = F(\delta - t_a) G(\delta - t_b) \quad (4)$$

Notice that the assumption that α and β are independent random variables enters here.

The probability on the right side of numerator of Eq. (3) is the time failure occurs. The distribution for this interval depends on the type of defect being modeled, but if we assume for this example that over a short interval (the interval of the Markov Chain) the time a defect causes failure is a continuous random variable with a constant rate of occurrence r, then

$$\{\delta \mid C_o^-\} = \begin{cases} r\ e^{-r(\delta - t_o)} & \text{for } \delta > t_o \\ o & \text{otherwise} \end{cases} \tag{5}$$

The rate r is an important variable that depends, for example, on the incidence of cracked components per time for a particular category of components (defined by age, material, loads worked, and other variables). It must be estimated from factory statistics on failures [2] [3].

Combining Eqs. (3), (4), and (5), we obtain

$$\{\delta \mid A_a^-, B_b^-, C_o^-\} = \frac{F(\delta - t_a)\ G(\delta - t_b)}{\int_{t_o}^{\infty} F(\delta - t_a)\ G(\delta - t_b)} \frac{r\ e^{-r(\delta - t_o)}}{r\ e^{-r(\delta - t_o)} d\delta} \tag{6}$$

And Eq. (1) becomes

$$\{A_o^+ \mid A_a^-, B_b^-, C_o^-\} = \frac{\int_{t_o}^{\infty} [1 - F(\delta - t_o)]\ F(\delta - t_a)}{\int_{t_o}^{\infty} F(\delta - t_a)\ G(\delta - t_b)} \frac{G(\delta - t_b)\, r\ e^{-r(\delta - t_o)} d\delta}{r\ e^{-r(\delta - t_o)}\ d\delta} \tag{7}$$

The lower limit of δ in the denominators of Eqs. (6) and (7) is determined by the inequality constraint in Eq. (5).

We define two intervals: (1) $I_a = t_o - t_a$, the interval from the time the system was last inspected (negatively) by Test A, and (2) $I_b = t_o - t_b$, the interval from the time the system was last inspected (negatively) by Test B. These intervals are shown in Figure 5. Using this notation and by changing variables in Eq. (7), we can write that

$$\{A_o^+ \mid A_a^-, B_b^-, C_o^-\} = \int_0^{\infty} [1 - F(\zeta)] F(I_a + \zeta) G(I_b + \zeta)\, r e^{-r\zeta} d\zeta \Big/$$

$$\int_0^{\infty} F(I_a + \zeta)\ G(I_b + \zeta)\ r\ e^{-r\zeta} d\zeta \tag{8}$$

The probability can be seen to be a function of

1) the rate at which defects occur - r;
2) the interval since the last negative inspection with Test A-I_a;
3) the interval since the last negative inspection with Test B-I_b;
4) the effectiveness of Tests A and B-$F(\alpha)$ and $G(\beta)$.

Other formulas are needed to complete the analysis. For example, if the inspection policy called for the application of Test B instead of Test A, then a formula for $\{B_o^+ \mid B_b^-, A_a^-, C_o^-\}$ is the mirror image of Eq.(8). Logic similar to that just shown can be used to derive a formula for the chance that, if Test A is negative at t_o, a defect will be discovered by Test B ($\{B_o^+ \mid A_o^-, B_b^-, C_o^-\}$). This is

$$\{B_o^+ \mid A_o^-, B_b^-, C_o^-\} = \frac{\int_0^{\infty} [1 - G(\zeta)]\ F(\zeta)\ G(\zeta + I_b)\ r\ e^{-r\zeta} d\zeta}{\int_0^{\infty} F(\zeta)\ G(\zeta + I_b)\ r\ e^{-r\zeta} d\zeta} \tag{9}$$

And if Tests A and B are both negative there is a chance that the component will cause a failure before the next scheduled inspection at time t_1, $\{C_1^+ \mid A_a^-, B_b^-, C_o^-\}$.

$$\{C_1^+ \mid A_a^-, B_b^-, C_o^-\} = \frac{\int_0^{I} F(\zeta + I_a)\ G(\zeta + I_b)\ r\ e^{-r\zeta} d\zeta}{\int_0^{\infty} F(\zeta + I_a)\ G(\zeta + I_b)\ r\ e^{-r\zeta} d\zeta} \tag{10}$$

where $I_a = t_o - t_a$, $I_b = t_o - t_b$, and $I = t_1 - t_o$. By letting either $t_a = t_b$, or $t_b = t_o$, or both t_a and $t_b = 0$, we can calculate the probabilitles $\{C_1^+ \mid A_o, B_b^-, C_o^-\}$, $\{C_1^+ \mid A_o^-, B_o^-, C_o^-\}$ and $\{C_1^+ \mid A_a^-, B_o^-, C_o^-\}$, which represent all the different inspection histories that can precede a failure.

The next step is to calculate the costs and benefits that arise from detecting some of the defects through tests before failure. This introduces the notion of the earliness of detection. Figure 6 shows the life of a particular defect as a time line. Markes on that line are four key points in the life of the defect: (from left to right) the time it first begins to form, the first time it is detectable by Test B(t_B), the first time it is detectable by Test A (t_A), and the time it will cause a system to fail (t_c). Suppose we performed both Tests A and B at time points t_{-2} and t_{-1}, and t_0 (now), as shown in the figure In this case no tests will detect the defect until the current inspection (at t_0), when Test B will be positive (since $t_0 > t_B$). Test A will be negative since $t_0 < t_A$. Now, the earliness of detection, which we designate as ε, is defined as the interval of time between the moment a defect is actually detected (t_0) and the moment it would have caused a failure in the absence of inspection (t_c). Thus in this case $\varepsilon = t_c - t_0$.

It should be obvious that the earliness of detection is a function of the effectiveness of the tests (greater effectiveness moves t_A and t_B to the left), and the frequency of screening (higher frequencies tend to move the time of actual detection (t_0) to the left). If we define ε_A as the earliness of defects detected by Test A, ε_B as the earliness of defects detectedby Test B, and ε_{AB} as the eariness of defects detected by Test B after Test A is negative, then it can be shown that

$$\{\varepsilon_A \mid A_0^+, A_a^-, B_b^-, C_0^-\} = k[F(\varepsilon+I_a)-F(\varepsilon)]\ G(\varepsilon+I_b)\ r\ e^{-r\varepsilon} \tag{11}$$

$$\{\varepsilon_B \mid B_0^+, B_b^-, A_a^-, C_0^-\} = k'[G(\varepsilon+I_b)-G(\varepsilon)]\ F(\varepsilon+I_a)\ r\ e^{-r\varepsilon} \tag{12}$$

$$\{\varepsilon_{AB} \mid B_0^+, A_0^-, B_b^-, C_0^-\} = k''\ F(\varepsilon)\ [G(\varepsilon+I_b)-G(\varepsilon)]\ r\ e^{-r\varepsilon} \tag{13}$$

Where k, k' and k'' are normalizing constants.

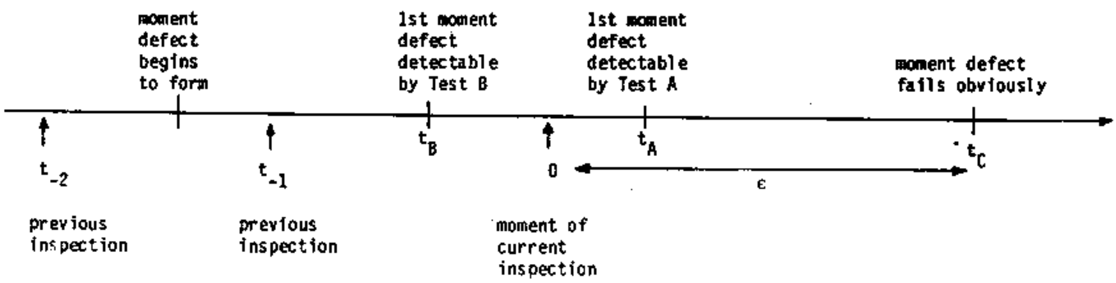

Fig. 6 . History of A Defect.

Formulas can be derived for the earliness of detection of defects detected by any means at a scheduled inspection with any policy. Notice that the earliness is a function of the age worked by the component since the most recent previous inspections by each test (I_A and I_B), as well as the effectiveness of the tests, as encoded in $F(\alpha)$ and $G(\beta)$.

Equations such as (8), (9) and (10) are used to calculate the chances that defects are detected through inspections. And Eqs. such as (11), (12) and (13) can be used to determine how early in their natural histories these defects are detected. Knowledge of the density functions for the earliness of detection of defects detected by various means is then used to calculate the future of those defects. In the Markov model this is handled by creating a set of earliness categories that collect defects that have similar earlinesses of detection. One can create as many earliness categories as are needed to model the problem realistically.

The last link that connects the detection of a defect to a set of outcomes is the outcome funtion. Here we assume that the future of a system with a defect is a function of how early in its history the defect was detected.

Suppose an important outcome is the failure of a system. An outcome function will relate the probability that a crack in a component of a size such that it will become obvious in x hours, will cause a failurs in y hours. More than one outcome can be analyzed by having more than one outcome function. For example, another outcome function can be used to relate the expected costs of repair to the earliness with which a crack is detected, while a third outcome function can simultaneously tally the probability that if unrepaired the defect will cause a failure in the future, and a fourth can compute the probability of a future failure given that the defect is repaired at the time of detection.

5. Data Needs of the Model

In order to implement the model, it is necessary to have specific entries for the functions that enter into the formulas just described. Three kinds of functions are required: (1) an incidence function that gives the probability that a defect causes a failure (in the absence of inspections) as a function of hours worked by the system since its last overhaul, (2) the density functions for the test intervals, and (3) the outcome functions.

There are many ways to estimate these functions, and the best method will depend on what data are available, what data are collectible, and what assumptions are appropriate. First, we expect that the incidence function can be estimated from data on the frequency of failures from defects in systems of various ages (i.e., hours worked since overhaul). It might be possible to use accident records to obtain the rate at which defects appear over a certain interval.

The second set of functions are the probability density functions for the test intervals. We can not observe these intervals directly, but they can be estimated from information about the proportion of defects detected by various methods (i.e., by various combinations of tests or through failure) in an experimentally scheduled inspection program. (for example, see Appendix For example, it may be possible to observe an experimental inspection program in which Tests A and B are performed every 100 hours on components with known defects. Then the proportions of defects revealed by Test A alone, by Test B alone, by both tests, and through failure, can be used to estimate the distributions on the two test intervals.

The estimation procedure is quite simple in concept: assume functional forms for the density function, then design or observe an experiment, write formulas that predict the outcomes of the experiment, and fit the parameters of the assumed functions to minimize the squared difference between the observed and predicted values. After the estimation procedure, it is necessary to test the closeness of the experimental frequencies to the theoretical frequencies, using the x^2 test.

The third set of functions is used to predict the outcomes associated with detecting defects by various methods before they cause a failure. These functions can be estimated by observing the outcomes that occur when each defect is detected by a particular method. We have seen that the way that a defect is detected (by a particular test or combination of tests, given a particular inspection history), implies a particular earliness of detection, which in turn implies a different outcome. For example, the cost of repairing a cracked component detected by one test may be less than that of a cracked component detected by another inspection if the former test can detect some cracks before they become obvious on latter inspection. The first component may only need trueing; the second may need to be replaced. To estimate the outcome functions records on the outcomes and costs associated with failures, accident reports, and maintenance records from the corresponding companies can be studied.

6. Conclusions

For any combination of operating conditions and inspection policies the model can calculate the chance that one or more defects will be detected at an inspection, the earliness with which a defect is detected, the outcomes (e.g., probability of failure, cost of repair) associated with detecting a defect, and the cost of inspections.

This general methodology can be applied to a wide variety of inspection problems such as the inspection of airplane, nuclear reactor, railcar, bridge and pipelines.

References

[1] Howard, R. A., Dynamic Probabilistic Systems, Vol. I & II, John Wiley & Sons, New York, 1971.

[2] Johnson, Duane P., Inspection Uncertainty; The Key Element in Nondestructive Inspection, Failure Analysis Associates, Palo Alto, May 1975.

[3] Smith, Stephan A. and Oren, Shamuel S., Reliability Growth of Repairable Systems, Analysis Research Group, Xerox Palo Alto Research Center, Palo Alto, 1979.

Appendix

An linear Approximate method of deriving probability distribution of the failure interval from small discrete data.

Let π be the failure interval between the detectable time, t_D, and the failure time, t_C, that is,

$$\pi = t_C - t_D.$$

At any relevent time, t_R, the critical crack sizes are considerably different, as shown in Figure A. 1.

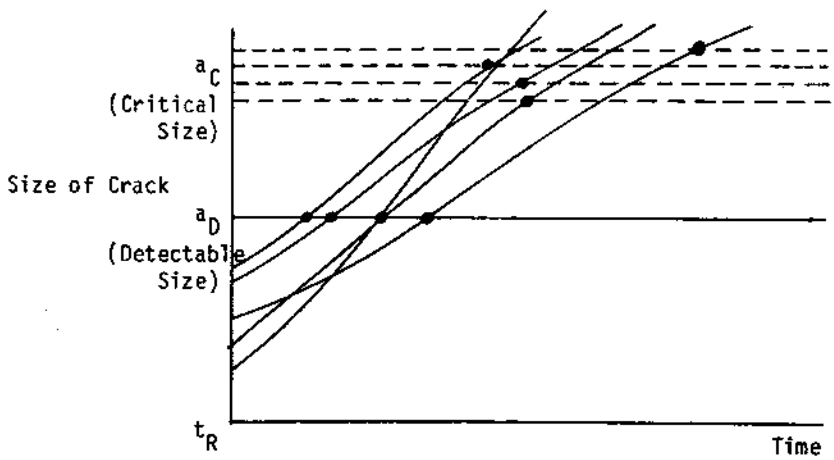

Fig. 1. Figure A.1. The growths of cracks

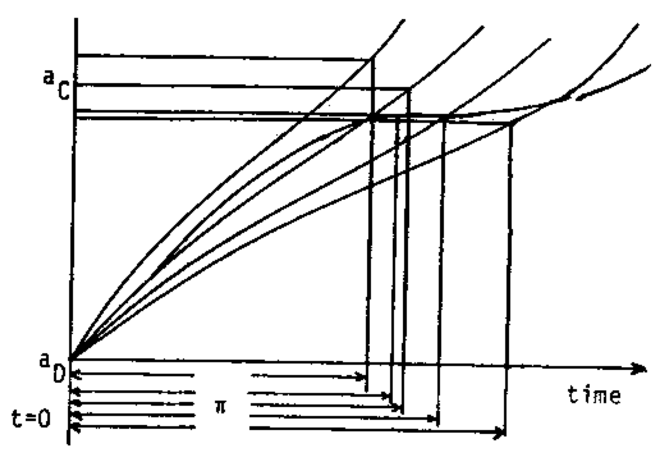

Figure A. 2. Crack growths at the detectable time.

In Figure A. 1, assume that detectable size only depends on inspection method, in other words, a_D has a constant value. The critical size, however, depends on the various internal and external effects of the corresponding component. That is, a_C has some probability distribution, $\{a_C\}$.

Redraw the previous figure as Figure A. 2.

In what follows, we confine our ourselves to study of linear growth. The general case of linear growth is

$$a(t) = \alpha t.$$

Where α (growth coefficient) is independent random variable.

We assume that α and crack size difference, M, between a_C and a_D follow normal distribution respectively. Then, a(t) has a normal distribution with the fo- llowing parameters.

$$\mu_{a(t)} = \mu_\alpha \cdot t \qquad \sigma^2_{a(t)} = \sigma^2_\alpha t^2.$$

And the survival probability is

$$\{\tau\rangle\ \pi\} = \{\alpha\pi \leqq M\} = \{\alpha\pi - M \leqq 0\}$$

Where τ is a life time.

Therefore, the failure probability is

$$\{\tau \langle\ \pi\} = 1 - \{\tau \geqq \pi\} = 1 - \{\alpha\pi - M \leqq 0\}$$

Let $y(\pi) = \alpha\pi - M$

then, $\mu_y = \mu_\alpha \pi - \mu_M \qquad \sigma^2_y = \sigma^2_\alpha \pi^2 + \sigma^2_M$

then, the failure probability is

$$\{\tau < \pi\} = 1 - \Phi\left(\frac{-(u_\alpha \pi - u_M)}{\sqrt{\sigma_\alpha \pi^2 + \sigma^2_M}}\right) = 1 - \Phi\left(\frac{-\pi + \frac{\mu_M}{\mu_\alpha}}{\sqrt{\frac{\sigma^2_\alpha \pi^2 + \sigma^2_M}{\mu^2_\alpha}}}\right)$$

$$= \Phi\left(\frac{\pi - \frac{\mu_M}{\mu_\alpha}}{\sqrt{\frac{\sigma^2_\alpha \pi^2 + \sigma^2_M}{\mu^2_\alpha}}}\right) \qquad (A.\ 1)$$

where $\Phi(\cdot)$ is culmulative standized normal dist'n.

The above dist'n involves three parameters;

$$a = \frac{\sigma^2_\alpha}{\mu^2_\alpha}, \quad b = \frac{\sigma^2_M}{\mu^2_\alpha}, \quad c = \frac{\mu_M}{\mu_\alpha} \qquad (A.\ 2)$$

Then, the dist'n takes the form

$$F(\pi) = \Phi\left[\frac{\pi - c}{\sqrt{a\pi^2 + b}}\right]$$

Based on the derived result, we can estimate the parameters of the distribution as follows:

1) Choose time points π_1, π_2 and π_3 such that $\pi_1 \langle\ \pi_2 \langle\ \pi_3$ and count the number of failures that lie in the in tervals $(0, \pi_j)$ for j= 1, 2 2, 3. Denote these numbers as $m(\pi_j)$. Calculate the ratios $\gamma(\pi_j) = \frac{m(\pi_j)}{N}$ where N is the total number of cracks.
2) Find the values $\Phi_j^{-1} = \Phi^{-1}(\gamma(\pi_j))$, where $\Phi^{-1}(\cdot)$ is inverse $\Phi(\cdot)$ distribution.
3) Use the method of successive approximations or a graphical method to solve the equation.

$$\frac{\pi_1 - \pi_2}{\pi_2 - \pi_3}\ \frac{\Phi_1^{-1}\sqrt{\frac{a}{b}\pi_1^2 + 1} - \Phi_2^{-1}\sqrt{\frac{a}{b}\pi_2^2 + 1}}{\Phi_2^{-1}\sqrt{\frac{a}{b}\pi_2^2 + 1} - \Phi_3^{-1}\sqrt{\frac{a}{b}\pi_3^2 + 1}}$$

in order to obtain $\frac{a}{b}$.

4) From the known ratio $\frac{a}{b} = d$, find

$$c = \frac{\pi_1(\pi_2 - c) - \pi_2(\pi_1 - c)}{(\pi_2 - c) - (\pi_1 - c)}\ \frac{\Phi_2^{-1}\pi_1\sqrt{d\pi_2^2 + 1} - \Phi_1^{-1}\pi_2\sqrt{d\pi_1^2 + 1}}{\Phi_2^{-1}\sqrt{d\pi_2^2 + 1} - \Phi_1^{-1}\sqrt{d\pi_1^2 + 1}}$$

5) Determine parpmeters b and a in Equation (A. 2) by using the following expression.

$$b = \left[\frac{\pi_1 - c}{\Phi_1^{-1}\sqrt{d\pi_1^2 + 1}}\right., \quad a = bd$$